Appendix

Problem 1

With two different and perfectly distinguishable demands for the final product and a fixed cost corresponding to a monopolically provided input, find the equilibrium prices and the number of firms in a monopolistic competition framework.

Let $K^H\left(P^H\right) = k_i^H + \sum_{j \neq i} k_j^H$ and $K^L\left(P^L\right) = k_i^L + \sum_{j \neq i} k_j^L$ be the different demands for the monopolistic competitors' product and P_X the fixed cost each of them faces. Each individual monopolistic competitor will solve the problem

(i) $$\max\ P^H \left[K^H\left(P^H\right) - \sum_{j \neq i} k_j^H \right] + P^L \left[K^L\left(P^L\right) - \sum_{j \neq i} k_j^L \right] - P_X$$

If there are n equal firms in the market, in equilibrium the following condition must apply

(ii) $$k_i^J = \frac{K^J}{n} \qquad i \qquad J$$

The first order conditions for the monopolistic competitor's problem can therefore be written as

(iii) $$\frac{K^H}{n} + P^H \frac{\partial K^H}{\partial P^H} = 0$$

(iv) $$\frac{K^L}{n} + P^L \frac{\partial K^L}{\partial P^L} = 0$$

From (iii) and (iv) it is possible to derive the equilibrium value for market demand elasticities

(v) $$\eta^H = \eta^L = \frac{1}{n}$$

which is the reciprocal of the number of firms.

To solve the problem, it would be necessary to find the equilibrium number of firms and substitute it in the expression for the equilibrium prices. To avoid excessive abstraction, we continue our analysis working with a linear-demand case.

The linear case

Let $K^H\left(P^H\right) = A^H - B^H P^H$ and $K^L\left(P^L\right) = A^L - B^L P^L$ be the demand functions. The first order conditions for the problem are

(vi) $$\frac{A^H - B^H P^H}{n} - B^H P^H = 0$$

(vii) $$\frac{A^L - B^L P^L}{n} - B^L P^L = 0$$

from where it is possible to obtain expressions for P^H and P^L as functions of n

(viii) $$P^H = \frac{1}{(n+1)} \frac{A^H}{B^H}$$

(ix) $$P^L = \frac{1}{(n+1)} \frac{A^L}{B^L}$$

and a price ratio which depends only on the parameters of the demand functions

(x) $$\frac{P^H}{P^L} = \frac{A^H / B^H}{A^L / B^L}$$

It is straightforward to see that a necessary and sufficient condition for a single price to hold is $\left(A^H / B^H\right) = \left(A^L / B^L\right)$

Substituting the results for P^H and P^L into the expressions for K^H and K^L respectively gives

(xi) $$K^H = A^H - \frac{A^H}{(n+1)} = \frac{n}{n+1} A^H$$

(xii) $$K^L = A^L - \frac{A^L}{(n+1)} = \frac{n}{n+1} A^L$$

Now we can replace equations (viii), (ix), (xi) and (xii) into the zero-profits condition

(xiii) $$\Pi = P^H K^H + P^L K^L - P_X = \frac{\left(A^H\right)^2}{B^H (n+1)^2} + \frac{\left(A^L\right)^2}{B^L (n+1)^2} - P_X = 0$$

where solving for n gives the equilibrium number of firms

(xiv) $$n = +\sqrt{\frac{\left(A^H\right)^2 / B^H + \left(A^L\right)^2 / B^L}{P_X}} - 1$$

and makes it possible to find the values for the equilibrium prices.

Consumer Surplus

In the linear demand case, consumer surplus is given by

(xv) $$V_1 = \frac{1}{2}\left[K^H\left(\frac{A^H}{B^H} - P^H\right) + K^L\left(\frac{A^L}{B^L} - P^L\right)\right]$$

Replacing (viii), (ix), (xi) and (xii) into the above expression yields

(xvi) $$V_1 = \frac{1}{2}\left[\frac{n}{(n+1)} A^H\left(\frac{A^H}{B^H} - \frac{1}{(n+1)}\frac{A^H}{B^H}\right) + \frac{n}{(n+1)} A^L\left(\frac{A^L}{B^L} - \frac{1}{(n+1)}\frac{A^L}{B^L}\right)\right]$$

which, after factorizing and rearranging, can be reduced to

(xvii) $$V_1 = \frac{1}{2}\left(\frac{n}{n+1}\right)^2\left[\frac{\left(A^H\right)^2}{B^H} + \frac{\left(A^L\right)^2}{B^L}\right]$$

<u>Problem 2</u>

With two different and perfectly distinguishable demands for the final product and a fixed cost corresponding to a monopolically provided input, find the equilibrium price and the number of firms in a monopolistic competition framework, given the constraint that each competitor must charge the same price to all consumers, regardless of their type.

Using the same notation as in the previous case, with the difference that now there will only be one price, P, the problem becomes

(xviii) $$\max\ P\left[K^H\left(P^H\right) - \sum_{j \neq i} k_j^H + K^L\left(P^L\right) - \sum_{j \neq i} k_j^L\right] - P_X$$

its first order condition being

(xix) $$\frac{K^H + K^L}{n} + P\left(\frac{\partial K^H}{\partial P} + \frac{\partial K^H}{\partial P}\right) = 0$$

The linear case

With the same functional forms as in problem 1, the first order condition becomes

$$\frac{\left(A^H + A^L\right) - P\left(B^H + B^L\right)}{n} - P\left(B^H + B^L\right) = 0 \tag{xx}$$

and the equilibrium price as a function of n takes the form of

$$P = \frac{1}{(n+1)} \frac{\left(A^H + A^L\right)}{\left(B^H + B^L\right)} \tag{xxi}$$

At this point, it is convenient to add horizontally both demands. Setting $A = A^H + A^L$, $B = B^H + B^L$ and $K = K^H + K^L$, the added-up demand function turns into $K = A - BP$. Inserting equation (xxi) into it gives

$$K = A - \frac{A}{(n+1)} = \frac{n}{n+1} A \tag{xxii}$$

Substituting equations (xxi) and (xxii) into the zero-profits condition

$$\Pi = \frac{1}{(n+1)^2} \frac{A^2}{B} - P_X = 0 \tag{xxiii}$$

allows to solve for n

$$n = +\frac{A}{\sqrt{BP_X}} - 1 = +\frac{A^H + A^L}{\sqrt{\left(B^H + B^L\right)P_X}} - 1 \tag{xxiv}$$

and consequently to find the value of P.

Consumer Surplus

In this problem, the linear-demand consumer surplus is given by

$$V_2 = \frac{1}{2}\left[K^H \left(\frac{A^H}{B^H} - P\right) + K^L \left(\frac{A^L}{B^L} - P\right)\right] \tag{xxv}$$

Replacing expression (xxi) into the demand functions allows to solve for K^H and K^L

(xxvi) $$K^H = A^H - \frac{1}{(n+1)}\frac{\left(A^H + A^L\right)}{\left(B^H + B^L\right)}B^H$$

(xxvii) $$K^L = A^L - \frac{1}{(n+1)}\frac{\left(A^H + A^L\right)}{\left(B^H + B^L\right)}B^L$$

Substituting expressions (xxi), (xxvi) and (xxvii) into (xxv) and remembering that $A = A^H + A^L$ and $B = B^H + B^L$ gives

(xxviii) $$V_2 = \frac{1}{2}\left\{\left[A^H - \frac{1}{(n+1)}\frac{A}{B}B^H\right]\left[\frac{A^H}{B^H} - \frac{1}{(n+1)}\frac{A}{B}\right] + \left[A^L - \frac{1}{(n+1)}\frac{A}{B}B^L\right]\left[\frac{A^L}{B^L} - \frac{1}{(n+1)}\frac{A}{B}\right]\right\}$$

which after some tedious algebra can be reduced to

(xxix) $$V_2 = \frac{1}{2}\left[\frac{\left(A^H\right)^2}{B^H} + \frac{\left(A^L\right)^2}{B^L} - \frac{(2n+1)}{(n+1)^2}\frac{A^2}{B}\right]$$

<u>Comparison between problems 1 and 2: equilibrium number of firms, equilibrium prices, consumer surplus and welfare</u>

Denoting with n_1 and n_2 the equilibrium number of firms in problems 1 and 2 respectively and using the same demand functions, a necessary and sufficient condition for n_1 to be greater than n_2 is

(xxx) $$\frac{\left(A^H\right)^2}{B^H} + \frac{\left(A^L\right)^2}{B^L} > \frac{\left(A^H\right)^2 + \left(A^L\right)^2 + 2A^H A^L}{B^H + B^L}$$

Adding up the right hand side and multiplying both sides by $\left(B^H B^L\right)\left(B^H + B^L\right)$

(xxxi) $$\left(A^H\right)^2 B^L\left(B^H + B^L\right) + \left(A^L\right)^2 B^H\left(B^H + B^L\right) > \left(A^H\right)^2\left(B^H B^L\right) + \left(A^L\right)^2\left(B^H B^L\right) + 2A^H A^L B^H B^L$$

Rearranging

(xxxii) $$\left(A^H B^L\right)^2 + \left(A^L B^H\right)^2 > 2\left(A^H B^L\right)\left(A^L B^H\right)$$

and factorizing

(xxxiii) $$\left(A^H B^L - A^L B^H\right)^2 > 0$$

which is true if and only if

(xxxiv) $$\frac{A^H}{B^H} \neq \frac{A^L}{B^L}$$

Equation (xxxiv) is false if and only if the price at which the quantity demanded is equal to zero is the same for both demand functions. In any other case, the equilibrium number of firms in the two price problem will be greater than that holding when only one price is allowed.

Note that if in equilibrium n_1 were equal to n_2, then P^H would be equal to P^L, and both would be equal to P, as the condition $\left(A^H/B^H\right) = \left(A^L/B^L\right)$ would hold. However, should n_1 be greater than n_2, P would take values between P^H and P^L, finding its equilibrium at a higher point than in the equal-number-of-firms solution. Figure i shows the solution for either problem when the condition $\left(A^H/B^H\right) = \left(A^L/B^L\right)$ holds.

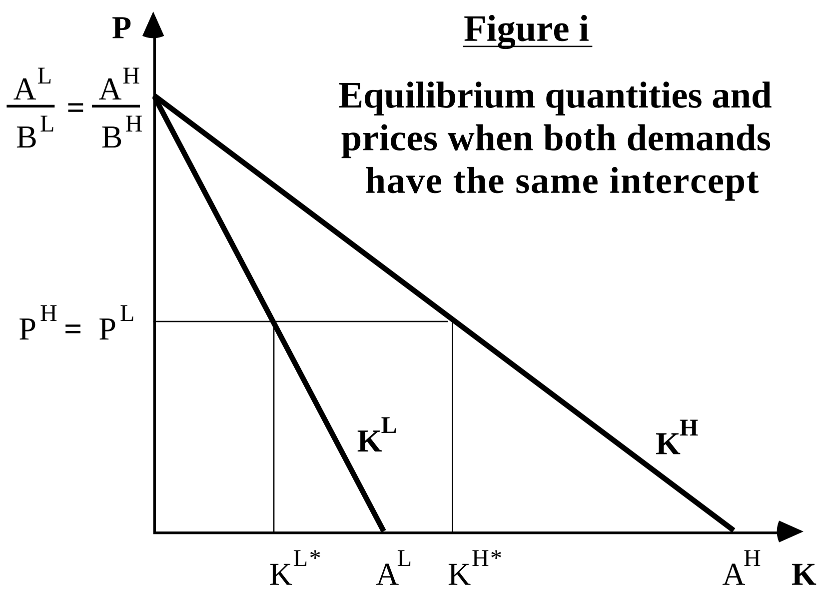

Figure i

Equilibrium quantities and prices when both demands have the same intercept

Let's now turn our attention to consumer surplus. Setting $Z = \frac{\left(A^H\right)^2}{B^H} + \frac{\left(A^L\right)^2}{B^L}$ and using (xiv), expression (xvii) can be rewritten as

(xxxv) $$2V_1 = \left(\frac{\sqrt{Z/P_X}-1}{\sqrt{Z/P_X}}\right)^2 Z = \left(1-\frac{1}{\sqrt{Z/P_X}}\right)^2 Z$$

and, after rearranging

(xxxvi) $$2V_1 = Z - 2\sqrt{ZP_X} + P_X$$

In a similar way, using (xxiv), it is possible to write (xxix) as

(xxxvii) $$2V_2 = Z - \left[\frac{\left(2A/\sqrt{BP_X}\right)-1}{\left(A/\sqrt{BP_X}\right)^2}\right]\left(\frac{A}{B}\right) = Z - \left(2A-\sqrt{BP_X}\right)\frac{\sqrt{BP_X}}{B}$$

and rearranging

(xxxviii) $$2V_2 = Z - 2A\sqrt{\frac{P_X}{B}} + P_X$$

Substracting (xxxviii) from (xxxvi)

(xxxix) $$2\left(V_1 - V_2\right) = 2\left(\frac{A}{\sqrt{B}} - \sqrt{Z}\right)\sqrt{P_X}$$

from where

(xl) $$\frac{V_1-V_2}{\sqrt{P_X}} = \frac{A^H + A^L}{\sqrt{B^H + B^L}} - \sqrt{\frac{\left(A^H\right)^2}{B^H} + \frac{\left(A^L\right)^2}{B^L}} = \left(n_2+1\right)\sqrt{P_X} - \left(n_1+1\right)\sqrt{P_X}$$

and finally

(xli) $$\frac{V_1-V_2}{P_X} = n_2 - n_1 \leq 0$$

Expression (xli) shows that, with the same demand functions, the two price solution always reduces consumer surplus respect to the one price solution, except for the case where the equilibrium number of firms is identical for both problems.

To analyze the change in welfare between the two price and the one price solutions we must also look at profits; as the monopolistic competitors always have zero profits, all benefits in the industry are those of the monopolist selling the input X. Assuming the monopolist's costs are fixed, the difference in his profits are equal to the difference in the equilibrium number of firms times the unit price of the input X. Therefore, the change in welfare between the solutions of problems 1 and 2 can be written as

(xlii) $$\Delta W = V_1 - V_2 + P_X\left(n_1 - n_2\right)$$

Dividing by P_X

(xliii) $$\frac{\Delta W}{P_X} = \frac{V_1 - V_2}{P_X} + n_1 - n_2$$

and using (xli)

(xliv) $$\Delta W = 0$$

Therefore, society does not gain nor lose from passing from the one price scheme to the two price scheme, as the loss in consumer surplus is exactly offset by the monopolist's increased profits.

For further clarification, a simple numerical example follows.

A numerical example

Let the demand functions be $K^H = 20 - 2P^H$ and $K^L = 10 - 2P^L$, and $P_X = 5$. The table gives the equilibrium values for prices, quantities and number of firms.

Variable	Two price case value	One price case value
n	6.07	5.7
P^H	1.41	-
P^L	0.7	-
P	-	1.12
K^H	17.18	17.76
K^L	8.59	7.76
V	92.13	93.95
W	122.5	122.5

Problem 3

With two different and perfectly distinguishable demands for the final product, different variable costs according to the type of customer to which the product is sold and a fixed cost, find the equilibrium prices and the number of firms in a monopolistic competition framework.

Keeping the usual conventions, the competitor's problem now can be expressed as

(xv) $$\max \left(P^H - P_X^H\right)\left[K^H\left(P^H\right) - \sum_{j\neq i} k_j^H\right] + \left(P^L - P_X^L\right)\left[K^L\left(P^L\right) - \sum_{j\neq i} k_j^L\right] - C$$

with C allowing for a residual fixed cost. The first order conditions become

(xvi) $$\frac{K^H}{n} + P^H \frac{\partial K^H}{\partial P^H} - P_X^H \frac{\partial K^H}{\partial P^H} = 0$$

(xvii) $$\frac{K^L}{n} + P^L \frac{\partial K^L}{\partial P^L} - P_X^L \frac{\partial K^L}{\partial P^L} = 0$$

As in the preceding problems, we will now go on with the linear case analysis

The linear case

Using the same demand functions as in problems 1 and 2, the first order conditions take the form of

(xviii) $$\frac{A^H - B^H P^H}{n} - B^H P^H + P_X^H B^H = 0$$

(xlix) $$\frac{A^L - B^L P^L}{n} - B^L P^L + P_X^L B^L = 0$$

from where prices as a function of the equilibrium number of firms may be obtained.

(l) $$P^H = \frac{1}{n+1}\left[\frac{A^H}{B^H} + nP_X^H\right]$$

(li) $$P^L = \frac{1}{n+1}\left[\frac{A^L}{B^L} + nP_X^L\right]$$

The price ratio may therefore be written as

(lii) $$\frac{P^H}{P^L} = \frac{A^H / B^H + nP_X^H}{A^L / B^L + nP_X^L}$$

Denoting with P_1^H and P_1^L the equilibrium prices resulting from problem 1 and with P_3^H and P_3^L those emerging from this problem, from equations (x) and (lii) it is possible to write

(liii) $$\frac{P_1^H}{P_1^L} = \frac{P_3^H}{P_3^L} \Leftrightarrow \frac{P_1^H}{P_1^L} = \frac{P_X^H}{P_X^L}$$

which says that the equilibrium price ratio in problems 1 and 3 will be the same if and only if the input price ratio in problem 3 is equal to the equilibrium price ratio in problem 1.

Using the expressions for prices in (l) and (li), quantities may be written as functions of n

(liv) $$K^H = A^H - \frac{A^H}{n+1} - \frac{n}{n+1} B^H P_X^H = \frac{n}{n+1}\left[A^H - B^H P_X^H\right]$$

(lv) $$K^L = A^L - \frac{A^L}{n+1} - \frac{n}{n+1} B^L P_X^L = \frac{n}{n+1}\left[A^L - B^L P_X^L\right]$$

Substituting (l), (li), (liv) and (lv) into the zero-profits condition gives

(lvi) $$\Pi = \frac{1}{n+1}\left[\frac{A^H}{B^H} + nP_X^H\right]\frac{n}{n+1}\left[A^H - B^H P_X^H\right] - P_X^H \frac{n}{n+1}\left[A^H - B^H P_X^H\right] + \\ + \frac{1}{n+1}\left[\frac{A^L}{B^L} + nP_X^L\right]\frac{n}{n+1}\left[A^L - B^L P_X^L\right] - P_X^L \frac{n}{n+1}\left[A^L - B^L P_X^L\right] - C = 0$$

and after rearranging

(lvii) $$\Pi = \frac{n}{(n+1)^2}\left\{\left[A^H - B^H P_X^H\right]\left[\frac{A^H}{B^H} - P_X^H\right] + \left[A^L - B^L P_X^L\right]\left[\frac{A^L}{B^L} - P_X^L\right]\right\} - C = 0$$

from where it is possible to solve for the equilibrium number of firms in terms of the parameters, and, substituting in equations (l) and (li), find the equilibrium prices. Further development of this analysis will not be conducted here, as no comparison between this problem and the preceding ones is possible with exogenously set input prices.