DESIGN OF FLYWHEELS FOR MOTOR-DRIVEN IMPULSE WHEELS

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The author investigates the problem of the design of flywheels for motor-driven impulse machines. He analyzes the characteristics of torque and speed curves of such machines and derives expressions by which the characteristics of these curves may be determined. A method of drawing such curves by means of a transmission dynamometer and steam-engine indicator is described. In appendices the author gives the derivation of the fundamental equations involved and of the working formulas which he presents, and makes calculations for an actual case. He also makes use of the Fourier development of the torque-time diagram as one method of solution.

CONSIDER an impulse machine, such as a punch or shear, which has most of the energy for its working stroke given to a flywheel by the motor during the interval between two strokes. The problem then arises as to what must be the proper size of flywheel to give out this energy during a drop in speed, which will not slow down the motor sufficiently to develop in it an excessive torque.

2 If all the energy used during the working stroke were given out by the flywheel the problem would be a simple one, but this is not the case. As the flywheel drops in speed, the motor does likewise, its torque increasing as its speed decreases. This varying torque integrated with respect to the angle turned through represents an amount of available work in addition to the energy given out by the flywheel.

3 At the conclusion of the working stroke another phase of the problem presents itself. There is a certain time interval before the machine will be required to make another stroke. During this interval the motor acts to accelerate the flywheel, its torque being a maximum at the beginning of the interval, and gradually decreasing as the speed builds up. What will be the speed when the next working stroke is about to be made?

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In actual industrial conditions it is very unlikely that there will be an unvarying interval between any two working strokes of such a machine. Consideration of this and the above facts will show that the initial speeds, speed drops, and torques of the different working strokes are by no means identical. It therefore becomes necessary to develop equations which will enable us to investigate the whole range of such possibilities for any combination of flywheel and motor.

Starting with the fundamental differential equation for this kind of motion, the author has, during the past two or three years studied various types of solutions for it, checking these with the flywheel and motor performances of a number of commercial machines. From the results of these comparisons he has been able to obtain formulas which are of a simple character and which have uniformly given accurate numerical results.

CHARACTERISTICS OF TORQUE AND SPEED CURVES

Before proceeding with the actual development of equations, it may serve to clear the ground if certain general characteristics of the torque and speed curves involved in the problem be indicated. (See Fig. 1.)

The resistance-torque curve consists of a series of peaks, representing the working strokes, connected by a (more or less) horizontal line representing the torque necessary to run the machine between strokes. The motor-torque curve is a wavelike line having its minimum value where it intersects the peak at the beginning of a working stroke, and its maximum where it intersects the peak near the end of a working stroke. This is obviously true, since these are the points where the motor-torque curve changes from being greater than the resistance torque curve to being less, and vice versa. From statements previously made it can be seen that these maxima and minima values can vary from peak to peak, depending upon the time dimension between them as indicated in the figure.

The height of the resistance curve is obviously the sum of the motor torque and the torque exerted by the flywheel, which latter is represented by the intercept between the two curves.

For the range in which we are interested, motor torque is sensibly proportional to motor slip. Slip in this discussion is understood to mean the actual difference between synchronous speed
and the speed under consideration at any instant. Consequently motor torque can be called —

$$Ks$$

where $s$ is the slip and $K$ is a factor or proportionality.

10 The torque exerted by the flywheel during a working stroke is —

$$J \frac{ds}{dt}$$

where $J$ is the mass moment of inertia of the flywheel, and $\frac{ds}{dt}$ is the angular deceleration.

11 From the above follows the differential equation —

$$J \frac{ds}{dt} + Ks = R \quad \ldots \ldots \ldots \ldots \quad [1]$$

$R$, the resistance torque, being some function of the time $t$. (See Appendix No. 1.)

12 It might be well at this point to transpose Equation [1] to

$$\frac{ds}{dt} = \frac{(R - Ks)}{J}$$

which is zero when $R = Ks$ or the two curves intersect. This agrees with previous statements as to the torque or slip being a maximum or minimum at these points.

13 The solution of Equation [1] is, of course, well known, provided that $R$ is known as a function of $t$. In general, however, it cannot be derived from known data, and by calculation based on experiment it can only be derived for the particular motor and
flywheel used. However, as will be shown later, motor-torque curves can be taken from a modified steam-engine indicator in conjunction with a transmission dynamometer. These can be developed by means of a Fourier series in such a way that the effect of motor and flywheel changes can be studied. (See Appendix No. 4.)

14 Curves made in this way also bring out the fact that the motor-torque curve very frequently conforms to a sine wave. On this assumption a study of Fig. 1 will show that the motor-torque curve is given by

\[ \text{Motor torque} = Ks' - Ka \cos qt \quad \ldots \ldots \quad [2] \]

where \( s' \) is the mean slip during the working stroke, \( q \) is the angular velocity of the imaginary vector generating the cosine function, and \( a \) is the amplitude of speed fluctuation above and below \( s' \).

From this

\[ s = s' - a \cos qt \]

\[ \frac{ds}{dt} = aq \sin qt \]

and Equation [1] can now be written as

\[ Jaq \sin qt + Ks' - Ka \cos qt = R \]

or

\[ Jaq \sin qt + Ks' - Ka \cos qt = Ks' + A \sin (qt - \epsilon) \quad [3] \]

where \( A = a(J^2q^2 + K^2)^{1/2} \) and \( \tan \epsilon = \frac{K}{Ja} \).

15 \( R \) and \( s \) are each expressed as a function of the time by two terms of a Fourier development. The \( R \) and \( s \) curves have a phase difference of \( \frac{\pi}{2} - \epsilon \). From Equation [3] formulas can be obtained (see Appendix No. 2), from which all necessary numerical results can be computed.

16 The above solution of the differential Equation [1] applies within the limits of the working stroke, that is to say, while the flywheel is decreasing in speed. We have now to consider what happens in the interval between two working strokes when the flywheel speed is increasing, and the resisting torque can be considered as constant, viz., the torque necessary for running the machine light. For these conditions,

\[- J \frac{ds}{dt} + Ks = Ks_0 \quad \ldots \ldots \ldots \ldots \quad [4]\]
s₀ being the slip at which the motor would run the machine light indefinitely. The solution of [4] is —

\[ t_a = \frac{J}{K} \log_e \left( \frac{s_1 - s_2}{s_2 - s_0} \right) \] .......................... [5]

or

\[ s_2 = e^{\frac{K}{J} t_a} (s_1 - s_0) + s_0 \] .......................... [6]

tₐ is the time between working strokes, s₁ the slip at the end of a working stroke, and s₂ the slip at the beginning of the next one.

17 From solutions [3], [5] and [6] Table 1 of formulas for practical calculation has been worked out. (See Appendix No. 2.) All these formulas are quite simple in character and provide means for estimating what may be expected to occur during a working stroke, whatever time may have elapsed since the preceding one.

HOW THE FORMULAS ARE APPLIED

18 We can now show how the formulas can be applied in practical calculation.

19 In any case we must know, to start with, the amount of work which the machine does in one stroke. If the machine exists only on paper, and no machine doing similar work is available, this must be estimated in the most convenient way. If, however, an existing machine be available for testing purposes, the amount of energy required per stroke can be found by a method described by the writer in Mechanical Engineering for July 1921.

20 From the product of this amount of energy and the number of working strokes per minute (plus a percentage depending upon the nature of the problem) the horsepower of the required motor can readily be determined. Its synchronous speed will depend upon the gear ratio and other design factors, into which it is unnecessary to enter here. At this point we have sufficient information to determine the quantities \( K, W, \) and \( \theta \) in Table 1.

21 As a first approximation, at any rate, \( w' \), the mean speed during the stroke, can be taken as the full-load speed. This fixes \( s' \), the mean slip, and also permits the computation of the quantity \( I \) (the torque-time integral or angular impulse) and \( t_w \) (the time for a working stroke).

22 We now come to the quantity \( E \). A glance at the diagram, Fig. 1, will show the meaning of this quantity better than it can be explained in words, but it might be described as the angle turned
through by the imaginary vector generating the resistance-torque curve while the curve is passing from the ordinate value $Ks_0$ to $Ks'$, that is, from light running resistance torque to resistance torque at mean slip.

23 This quantity $E$ can now be computed by the short formula

$$E = \frac{\tan E}{2} \tan 15^\circ$$

given in the table. If most of the work during the working stroke be done by the flywheel, as is generally the case, $E$ will come out less than 15 deg. and be quite accurate enough. It sometimes happens, however, in the case of small peak loads of frequent occur-

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**TABLE 1 FLYWHEEL AND MOTOR FORMULAS**

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>DESCRIPTION</th>
<th>UNIT</th>
<th>FORMULA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>Amplitude of resistance-torque curve</td>
<td>Pound-feet</td>
<td>$\frac{K(s' - s_0)}{\sin E}$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Amplitude of speed fluctuation</td>
<td>Radians per sec.</td>
<td>$\frac{A}{\sqrt{J^2 \omega^2 + K^2}}$</td>
</tr>
<tr>
<td>$1E$</td>
<td>Angular lag of $A$-curve from $s = s_0$</td>
<td>Radians</td>
<td>$\sqrt{\frac{Ks_0(s' - s_0)}{1 - Ks_0} + \frac{\pi^2}{4}}$</td>
</tr>
<tr>
<td>$1$</td>
<td>Angular advance of motor-torque curve over $A$-curve</td>
<td>Radians</td>
<td>$\tan \epsilon = \frac{K}{JQ}$</td>
</tr>
<tr>
<td>$J$</td>
<td>Mass moment of inertia of flywheel on motor shaft</td>
<td>Ft-lb.sec.$^2$</td>
<td>$\frac{W}{\theta}$ or $\omega^2$, or $\frac{1}{q} \sqrt{\left(\frac{a + K}{a} - K\right)\left(\frac{a}{a} - K\right)}$</td>
</tr>
<tr>
<td>$K$</td>
<td>Factor of proportionality of slip to motor torque</td>
<td>Ft-lb-sec.$^2$</td>
<td>$\frac{\pi + 2E}{t_w}$</td>
</tr>
<tr>
<td>$q$</td>
<td>Angular velocity of radii of circles generating curves</td>
<td>Radians per sec.</td>
<td>$\frac{1}{q} \sqrt{\left(\frac{a + K}{a} - K\right)\left(\frac{a}{a} - K\right)}$</td>
</tr>
<tr>
<td>$s'$</td>
<td>Mean slip during stroke</td>
<td>Rad. per sec.</td>
<td>$\frac{Ks_0}{s_1 + s_0}$</td>
</tr>
<tr>
<td>$s_0$</td>
<td>Light running slip</td>
<td>Rad. per sec.</td>
<td>$\frac{\epsilon}{\epsilon - (s' - s_0) + s_0}$</td>
</tr>
<tr>
<td>$s_1$</td>
<td>Slip at end of stroke</td>
<td>Rad. per sec.</td>
<td>$\frac{J}{K} \log_\epsilon \left(\frac{s_1}{s_1 - s_0} \right)$</td>
</tr>
<tr>
<td>$T$</td>
<td>Torque at any instant</td>
<td>Pound-feet</td>
<td>$\frac{1}{\theta} \log_\epsilon \left(\frac{s_1}{s_1 - s_0} \right)$</td>
</tr>
<tr>
<td>$t_w$</td>
<td>Time for flywheel to regain speed</td>
<td>Seconds</td>
<td>$\frac{1}{\theta} \log_\epsilon \left(\frac{s_1}{s_1 - s_0} \right)$</td>
</tr>
<tr>
<td>$W$</td>
<td>Work per stroke</td>
<td>Foot-pounds</td>
<td>$\frac{1}{\theta} \log_\epsilon \left(\frac{s_1}{s_1 - s_0} \right)$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Angular displacement of motor shaft during stroke</td>
<td>Radians</td>
<td>$60 \frac{1}{\theta}$</td>
</tr>
<tr>
<td>$\omega'$</td>
<td>Mean speed during stroke</td>
<td>Rad. per sec.</td>
<td>$\frac{1}{\theta} \log_\epsilon \left(\frac{s_1}{s_1 - s_0} \right)$</td>
</tr>
</tbody>
</table>

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1 $F(E) = (\pi + 2E) \tan E - \frac{2Ks_0(s' - s_0)}{1 - Ks_0} = 0$ Use if $E > 15^\circ$ $F(E) = 2 \tan E + \frac{\pi + 2E}{\cos E}$
rence, that a powerful motor is used in conjunction with a light flywheel, when $E$ may be considerably greater than 15 deg. In these cases $E$ must be computed by the longer method shown in Appendix No. 2.

24 Knowing $E$, we can compute $A$, and then have the resistance torque as a function of the time, expressed by the first two terms of a Fourier development. It might be stated here that the great majority of impulse machines, whose work consists of punching, shearing, applying pressure, etc., actually show resistance curves whose shape can be closely approximated by a function of this general character. In cases where this is not sufficiently close, it becomes necessary either to use more terms of a Fourier development or to proceed by the "short arc" method, as described in Appendix No. 4. This involves more work, especially if the stroke is to be studied over any considerable range, and it is not often necessary as the drop in speed of a flywheel depends much more upon the diagram's area than upon its shape.

25 The quantity $a$, or the amplitude of speed fluctuation, can be assumed according to the number of working strokes per minute and the allowable momentary overload of the motor. This completes all the values necessary for the calculation of $J$, the mass moment of inertia of the flywheel.

26 For the first stroke $a$ can be taken as $s' - s_0$, in which case $E$ and $e$ are identical. For the second stroke we can assume a slightly greater value for $s'$ and therefore for $w'$, and compute $t_w$, $I$, $E$, $q$, $e$, and $A$ to correspond. $J$, of course, is the same as before, so we must calculate another $a$ from the table; $s_2$ must now be computed as the slip at the beginning of the second stroke. We can now compute $t_w$, using this value of $s_2$ and of course the $s_1$ of the first stroke.

27 Assuming still greater values of $s'$ for successive strokes, and proceeding in like manner, we can compute a table as shown in Appendix No. 3, giving speed drops, torques, etc., for strokes made with all kinds of time intervals between them. In this way we can investigate, as fully as desirable, all the conditions under which the motor may have to perform.
APPENDIX NO. 1
DERIVATION OF FUNDAMENTAL EQUATIONS

28 Consider a flywheel and motor set. This can be regarded as a conservative system having a potential function. When kinetic energy is taken from the flywheel by the periodic load, the drop in speed causes an increased torque in the motor. When the load period comes to an end, the motor torque is usually just past its maximum value, and the work which it does in falling back to the torque at the commencement of the next stroke, is employed in increasing the speed of the rotating parts and thus is converted into kinetic energy. In other words, the kinetic energy lost by the flywheel during a stroke can be regarded as though it were stored up in the motor in the form of potential energy. This is really analogous to the transformation of energy which is continually taking place from kinetic to potential and vice versa in such a device as a pendulum. The only difference is that the “potential” function is electrical instead of being gravitational.

29 Consider Lagrange’s equation for motion of this kind, with the usual notation —

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \]

where \( L \) is kinetic energy - potential energy.

30 Let \( T \) = motor torque. The “potential” energy is then \( \int (R - T) \, d\theta \), that is, the energy given out by the flywheel, and, according to the above conception, “stored up” in the motor. The kinetic energy is \( \frac{1}{2} J w^2 \). \( \theta \) corresponds to \( q \) and \( w \) to \( \dot{q} \).

Then —

\[ L = \frac{1}{2} J w^2 - \int (R - T) \, d\theta \]

\[ \frac{\partial L}{\partial q} \quad or \quad \frac{\partial L}{\partial \dot{q}} = J w \quad and \quad \frac{d}{dt} \left( J w \right) = J \frac{dw}{dt} \]

Since \( w = w_1 - s \), where \( w_1 \) = synchronous speed and \( s \) = slip,

\[ J \frac{dw}{dt} = \frac{d}{dt} \left( J w_1 - J s \right) = -J \frac{ds}{dt} \]

Also —

\[ \frac{\partial L}{\partial q} \quad or \quad \frac{\partial L}{\partial \theta} = -(R - T) \]

so that Lagrange’s equation becomes —

\[ J \frac{ds}{dt} + T = R \]

or, if torque be proportional to slip for the range under consideration, \( T = K s \) and we get —

\[ J \frac{ds}{dt} + K s = R \]
This is the differential equation for the working stroke.

31 For the period of acceleration between strokes the "potential" function is

\[ \mathcal{J}(T - R) \, d\theta \]

and the kinetic energy is the same as before or \( \frac{1}{2}Jw^2 \), so that \( L \) becomes

\[ \frac{1}{2}Jw^2 - \mathcal{J}(T - R) \, d\theta \]

and Lagrange's equation reduces to

\[-J \frac{ds}{dt} + T - R = 0\]

for the period between strokes. The quantity \( q \) in the above, which is a generalized coordinate of Lagrange's equation, has nothing to do with the \( q \) representing radians per second which appears elsewhere.

**APPENDIX NO. 2**

**DERIVATION OF WORKING FORMULAS GIVEN IN TABLE 1**

32 The diagram of Fig. 1 is a graphic representation of Equation [2] in Par. 14. Vertical ordinates represent torque and horizontal ordinates represent time multiplied by the quantity \( q \), which is the (constant) angular velocity of the imaginary vectors generating the sine curves. The zero is chosen at the point where the motor-torque curve at its minimum cuts the resistance-torque curve, and this gives the intersection of the two curves for maximum motor torque at the ordinate corresponding to \( qt = \pi \). \( E \) is the angular displacement from the point where the resistance torque begins to the ordinate of average height of the motor-torque curve, and \( \varepsilon \) is the phase difference between the motor-torque curve and the resistance-torque curve, less 90 deg.

33 The angular impulse (or \( \mathcal{J}Rdt \) integral) for the motor shaft during a working stroke is given by

\[ \mathcal{J} \left[ A \sin(qt - E) + Ks' \right] dt \]

with the proper limits inserted. With the zero point as above these limits are from \((\pi + 2E)/q\) to 0, so that we have

\[ \text{Angular impulse} = I = \frac{1}{q} \left[ 2A \cos E + Ks'(\pi + 2E) \right] \ldots \ldots \ldots \text{[7]} \]

34 If the time for the whole working stroke be \( t_w \), then \( qt_w = \pi + 2E \) and

\[ q = \frac{(\pi + 2E)}{t_w} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \text{[8]} \]

35 From [7] and [8],

\[ I(\pi + 2E) = t_w[2A \cos E + Ks'(\pi + 2E)] \]
and—

\[ A = \frac{(I - Ks't_w)}{2t_w \cos E} \]  \hspace{1cm} [9]

36 From the diagram,

\[ A = \frac{K(s' - s_0)}{\sin E} \]  \hspace{1cm} [10]

and from [9] and [10],

\[(\pi + 2E) \tan E = \frac{2t_wK(s' - s_0)}{I - Ks't_w} \]  \hspace{1cm} [11]

37 If \( E \) be small, so that \( \tan E \) is approximately equal to \( E \), [11] reduces to—

\[ E = \sqrt{\frac{K(s' - s_0)t_w}{I - Ks't_w}} + 0.615 - 0.7854 \]

but if \( E \) comes out with a value greater than about 15 deg., then Equation [11] must be solved as it stands. This can be done most readily by writing—

\[ f(E) = (\pi + 2E) \tan E - \frac{2Kt_w(s' - s_0)}{I - Ks't_w} = 0 \]  \hspace{1cm} [12]

\[ f'(E) = 2 \tan E + \frac{\pi + 2E}{\cos^2 E} \]

then if a trial value of \( E \), say \( E' \), be taken,

\[ E' = \frac{f(E')}{f'(E')} \]

will be a closer approximation to that value of \( E \), which satisfies Equation [12]. The process can of course be carried on to obtain any desired degree of accuracy.

38 From Equation [3], Par. 14, \( A \) is seen to be \( a \sqrt{J^2q^2 + K^2} \), from which

\[ J = \frac{1}{q} \sqrt{\left( \frac{A}{a + K} \right) \left( \frac{A}{a - K} \right) } \]

and \( s_1 \) the slip at the end of a working stroke is very evidently \( s' - a \times \cos (\pi + E + \epsilon) \), or \( s' + a \cos (E + \epsilon) \).

APPENDIX NO. 3

CALCULATION OF A PRACTICAL CASE

39 Consider a machine which absorbs 2000 ft-lb. of energy per stroke and which may operate often enough to require a 2-hp. motor. Synchronous speed, 1800 r.p.m.; full-load speed, 1730 r.p.m.

\[ K = \frac{50,150 \times 2}{70 \times 1,730} = 0.83 \]
40 Suppose the gear ratio from motor shaft to operating shaft to be 10 to 1, and the working stroke to be done in one revolution of the operating shaft; then —

\[ \theta = 62 \text{ radians} \]

\[ w' = \frac{1,730 \times 3.14}{30} = 181 \text{ radians per sec.} \]

\[ s' = \frac{(70 \times 3.14)}{30} = 7.3 \text{ radians per sec.} \]

Assume —

\[ s_0 = 2 \text{ radians per sec.} \]

\[ t_o = \frac{62}{181} = 0.343 \text{ sec.} \]

\[ I = \frac{2,000}{181} = 11.05 \text{ ft-lb-sec.} \]

\[ E = \sqrt{\frac{0.83 \times 5.3 \times 0.343}{11.05 - (0.83 \times 0.343 \times 7.3)} + 0.615 - 0.7854} = 0.1 \text{ radian or 5.73 deg.} \]

The value of \( E \) is small enough to be within the limits of accuracy of this formula.

\[ A = \frac{(0.83 \times 5.3)}{\sin 5.73^\circ} = 44 \text{ lb-ft.} \]

\[ q = \frac{3.14 + 0.2}{0.343} = 9.72 \text{ radians per sec.} \]

If we now consider the amplitude \( a \) to be 5.3, that is, for the flywheel to be at light running speed when the working stroke begins, then \( A = 44/5.3 = 8.3 \) and \( J \), the mass moment of inertia of the flywheel,\(^1\) is —

\[ J = \frac{\sqrt{(8.3 + 0.83)(8.3 - 0.83)}}{9.72} = \frac{0.85 \text{ ft}^2\text{-lb.}}{32.2} \]

41 If we now take \( s' \) for the second stroke as 10 radians per sec., we find that \( t_o = 0.348 \text{ sec.}, I = 11.25 \text{ ft-lb-sec.}, E = 0.16 \text{ radian or 9.16 deg.}, q = 9.95 \text{ radians per sec.}, \) and \( A = 41.5 \text{ lb-ft.} \) \( J \) is still 0.85 and therefore \( a \) becomes —

\[ \frac{41.5}{\sqrt{(0.85 \times 9.95)^2 + 0.83^2}} = 4.88 \text{ radians per sec.} \]

\( e \) for this stroke is —

\[ \tan^{-1} \left[ \frac{0.83}{0.85 \times 9.95} \right] = 0.098 \text{ radian} \]

and \( E - e = (0.16 - 0.098) = 0.062, \) so that \( s_2 \) the slip at the beginning of this stroke is —

\[ 10 - [4.88 \times \cos (0.062 \text{ radian})] = 5.12 \text{ radians per sec.} \]

and \( s, \) the slip at the end of the first working stroke is found to be \( 8.3 + [5.3 \cos (0.2 \text{ radian})] \) or 13.5 radian per sec., so that \( t_o, \) the time interval between the strokes, is —

\[ t_o = \frac{0.85}{0.83} \times \log_e \left( \frac{13.5 - 2}{5.12 - 2} \right) = 1.34 \text{ sec.} \]

\( ^1 \) This is the mass moment of inertia of a flywheel on the motor shaft. For the same flywheel effect on any other shaft, the mass moment of inertia varies as the square of the speed ratio between the shafts.
Proceeding in this way we can tabulate results as below:

<table>
<thead>
<tr>
<th>Stroke No.</th>
<th>A</th>
<th>a</th>
<th>E</th>
<th>s</th>
<th>l</th>
<th>q</th>
<th>s'</th>
<th>a1</th>
<th>s2</th>
<th>t0</th>
<th>t2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>44.0</td>
<td>5.3</td>
<td>0.10</td>
<td>0.10</td>
<td>11.05</td>
<td>0.72</td>
<td>7.3</td>
<td>13.5</td>
<td>2</td>
<td>1.34</td>
<td>0.243</td>
</tr>
<tr>
<td>2</td>
<td>41.5</td>
<td>4.91</td>
<td>0.16</td>
<td>0.098</td>
<td>11.25</td>
<td>0.95</td>
<td>10.0</td>
<td>14.75</td>
<td>5.1</td>
<td>0.348</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>40.0</td>
<td>4.65</td>
<td>0.207</td>
<td>0.091</td>
<td>11.35</td>
<td>10.10</td>
<td>12.0</td>
<td>16.46</td>
<td>7.4</td>
<td>0.88</td>
<td>0.352</td>
</tr>
</tbody>
</table>

which will enable us to study the effect of working strokes beginning at any speed of the motor, or after any time interval between strokes.

APPENDIX NO. 4

DEVELOPMENT OF THE DIFFERENTIAL EQUATION \( \frac{d^2s}{dt^2} + Ks = R \)

FOR THE GENERAL CASE WHERE \( R = f(t) \)

43 Rearranging the above equation, we have —

\[
\frac{ds}{dt} + \frac{Ks}{J} = \frac{R}{J}
\]

\[\text{[13]}\]

44 \( R \) is to be developed as a Fourier series in the form —

\[A_0 + A_1 \sin qt + B_1 \cos qt + A_2 \sin 2qt + B_2 \cos 2qt \ldots \]

or, more briefly,

\[f(t) = A_0 + \sum_{n=1}^{\infty} A_n \sin nqt + B_n \cos nqt \]

45 \( A_0 \) is the average height of the curve, or the average torque, or \( K \) multiplied by the average slip \( = K_0 \). Equation [13] now becomes —

\[
\frac{ds}{dt} + \frac{Ks}{J} = \frac{K_0}{J} + \frac{1}{J} \sum_{n=1}^{\infty} A_n \sin nqt + B_n \cos nqt
\]

the solution of which is —

\[s = \epsilon \left( -\frac{K}{J^t} \right) \left\{ \int \left( A_n \sin nqt + B_n \cos nqt \right) dt + C \right\}
\]

Integrating,

\[s = \epsilon \left( -\frac{K}{J^t} \right) \left\{ s_0 \epsilon^\frac{K}{J} \right\} + \frac{K}{J} \sum A_n (\frac{K_0}{J} \sin nqt - nq \cos nqt) + B_n (\frac{K_0}{J} \cos nqt + nq \sin nqt) + C \]

\[+ \frac{K}{J^t} \left( \frac{K^2}{J^2 + n^2q^2} \right) \]
Now \( \frac{K}{J} \sin nqt - nq \cos nqt = \sqrt{\frac{K^2}{J^2} + n^2q^2} \sin (nqt - \epsilon_n) \)

and \( \frac{K}{J} \cos nqt + nq \sin nqt = \sqrt{\frac{K^2}{J^2} + n^2q^2} \sin (nqt + \epsilon_n) \)

where \( \cot \epsilon_n = \tan \epsilon_n = \frac{K}{Jq} \) and \( e^{\frac{K}{J}t} - e^{-\frac{K}{J}t} = 1 \), so that the above equation simplifies to

\[
\delta = \delta_a + \sum_{1}^{\infty} \frac{A_n \sin (nqt - \epsilon_n) + B_n \sin (nqt + \epsilon_n)}{\sqrt{K^2 + J^2 n^2q^2}} + C \epsilon^{\frac{K}{J}t} \ldots \quad [13a]
\]

46 Let \( s = s_0 \) when \( t = 0 \) at the beginning of a working period, and we get

\[
s_0 = s_a + \sum_{1}^{\infty} \frac{B_n \sin E_n - A_n \sin \epsilon_n}{\sqrt{K^2 + J^2 n^2q^2}} + C^* \]

where \( C^* \epsilon^{\frac{K}{J}t} = 0 = 1 \).

Finding \( C \) from this and inserting this value in Equation \([13a]\), we have

\[
\delta - \delta_0 = (\delta_a - \delta_0) \left( 1 - \epsilon^{\frac{K}{J}t} \right) - \epsilon^{\frac{K}{J}t} \sum_{1}^{\infty} \frac{B_n \sin E_n - A_n \sin \epsilon_n}{\sqrt{K^2 + J^2 n^2q^2}}\]

\[
+ \sum_{1}^{\infty} \frac{A_n \sin (nqt - \epsilon_n) + B_n \sin (nqt + \epsilon_n)}{\sqrt{K^2 + J^2 n^2q^2}} \ldots \ldots \ldots \quad [14]
\]

47 This is the general solution for \( s = s_0 \), or the drop in speed at any time \( t \) below the speed at the commencement of the working period. To calculate the actual values we must know all the coefficients \( A_n \) and \( B_n \) of the Fourier series.

48 When the motor has been running for some time, or \( t \) has become large, \( \epsilon^{\frac{K}{J}t} \) approaches zero, and Equation \([14]\) above approaches the form

\[
s = s_a + \sum_{1}^{\infty} \frac{A_n \sin (nqt - \epsilon_n) + B_n \sin (nqt + \epsilon_n)}{\sqrt{K^2 + J^2 n^2q^2}} \ldots \ldots \ldots \quad [15]
\]

49 This transformation is due to the fact that, as time goes on, the terms containing \( \epsilon^{\frac{K}{J}t} \) tend to "damp" themselves out of existence. In most practical cases \( \epsilon^{\frac{K}{J}t} \) would be negligible at the end of a very few minutes' running of a machine.
50 It might be noted here that if it be accurate enough to express the resistance torque by a simple sine function of the time, $A \sin qt$, then [15] reduces to

$$\begin{align*}
s &= s_a + \frac{A_1 \sin (qt - \epsilon)}{\sqrt{K^2 + J^2q^2}} \\
\text{s}_{\text{max}} &= s_a + \frac{A_1}{\sqrt{K^2 + J^2q^2}}
\end{align*}$$

and, by giving $\epsilon$ the proper value, $s$ would become $s_a - a \cos qt$, $a$ being $\frac{A}{\sqrt{K^2 + J^2q^2}}$ corresponding with the formulas given in the working table.

51 To get the coefficients $A_n$ and $B_n$ suppose we have a diagram showing torque and time. The author has obtained many of these by means of a transmission dynamometer fitted with an ordinary steam-engine indicator, whose pencil marks on a strip of paper moving at a constant speed. Such an arrangement is shown in Fig. 2. The driven machine has some temporary motor and flywheel which will make it run. We therefore know $J$, and the periodic time $t$, can be found from the diagram. $q = \frac{2\pi}{t}$; $K$ is a characteristic of the motor, while $e_n$ for each term can be found from $J$, $q$, $n$ and $K$. $s_a$ is the average height of the diagram divided by $K$, and from the properties of a Fourier series,

$$\begin{align*}
A_n &= \frac{2\sqrt{K^2 + J^2n^2q^2}}{t} \int_0^{t_1} s \sin (nt - e_n)dt \\
B_n &= \frac{2\sqrt{K^2 + J^2n^2q^2}}{t} \int_0^{t_1} s \cos (nt - e_n)dt
\end{align*}$$

$s$ being of course the torque anywhere divided by $K$. The constants $A_n$ and $B_n$ can now be developed by the application of the usual analytic and graphic methods, the graphic method being specially recommended for the development of arbitrary functions. It is now possible to plot the slip curves of other combinations of flywheel and motor by substituting the proper values of $J$ and $K$.

52 There remains another method of studying motor and flywheel effects on a machine. By driving the strip of paper past the indicator pencil from some convenient shaft of the dynamometer, we can get a diagram giving torque and angular displacement. The equation

$$J \frac{ds}{dt} + Ks = R$$

can then be written

$$J \omega \frac{ds}{d\theta} + Ks = R$$

$R$ being resistance torque and $\omega$ angular velocity. $J$ must be assumed, $s$, as before, is torque divided by $K$, and $\omega = \omega' - s$, where $\omega'$ is initial angular velocity.
To plot the R-curve it is necessary to know $\frac{ds}{d\theta}$ to do which there is a choice of the following methods:

1. Draw many equidistant ordinates and divide the change in torque by the change in angle and by $K$

2. Develop the curve in the form of a Fourier series, $Ks = A_0 + A_1 \sin p\theta + B_1 \cos p\theta + A_2 \sin 2p\theta$, etc., from which

$$\frac{ds}{d\theta} = \frac{1}{K} \left[ A_1 p \cos p\theta - B_1 p \sin p\theta, \text{etc.} \right]$$

3. Use a mechanical differentiator.
In any case, supposing the \( R \)-curve to have been obtained, we can now solve the equation

\[
\frac{ds}{d\theta} = \frac{R - T}{Jw}
\]

and proceed as follows:

Let \( \delta \theta \) be a small increment of angular displacement, the smaller the more accurate. We know or can assume the value \( s \) of the slip at the beginning of the working stroke, and also \( R \) and \( w \). A trial value of \( J \) can be assumed. Then \( \frac{ds}{d\theta} \) can be calculated and also \( \delta s \), a small increment in the torque, since

\[
\delta s = \delta \theta \left( \frac{ds}{d\theta} \right), \quad w = w' - s, \quad \text{and} \quad T, \quad \text{the effort torque, can be taken as } Ks, \quad \text{or can be read from the torque-speed performance chart of the proposed motor. The small portion of a table below will show the method more easily than it can be explained. Take } J \text{ as } 0.5 \text{ ft-lb.}
\]

Other units are in feet, pounds, radians and seconds.

<table>
<thead>
<tr>
<th>( \delta \theta )</th>
<th>( \frac{ds}{d\theta} )</th>
<th>( \delta s )</th>
<th>( R )</th>
<th>( T )</th>
<th>( s )</th>
<th>( w )</th>
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<td>41</td>
<td>5.4</td>
<td>1.70</td>
<td>124</td>
</tr>
</tbody>
</table>

Starting with \( R = 3, \ T = 2, \ w = 125, \ s = 1 \),

\[
\frac{ds}{d\theta} = \frac{3 - 2}{0.5 \times 125}
\]

or 0.016. The increment \( \delta \theta \) having been taken as 1 for simplicity, \( \delta s = 1 \times 0.016 \) or 0.016. \( w = 125 - 0.016 \) or almost no change. The new value of \( s \) is 1 plus the increment 0.016 or 1.02 in round figures, which corresponds to a torque, taken from a motor torque-speed chart, of 4 lb-ft. The next value of \( \frac{ds}{d\theta} \) is \( \frac{10 - 4}{0.5 \times 125} \) or 0.096. The process can now be continued through one or more complete cycles of the machine's action.

**DISCUSSION**

**Alphonse A. Adler** asked if the author's formula could be used for conditions involving direct-current motor drive. Also what
DISCUSSION

modifications, if any, could be made in the relation of torque to drop in speed in direct-current motor drives so that Lagrange's equation might be applicable.

The Author. The formula in question can be used for any form of motor provided that the torque varies directly as the slip, in other words, as long as the relation

$$T = Ks$$

holds. If the torque is some other function of the slip or, in fact any function of the slip, then Lagrange's equation, as shown in Par. 30, becomes,

$$J \cdot \frac{ds}{dt} + T = R$$

$T$ and $R$ being any functions of the time. This can be thrown into the form

$$J \cdot \frac{w \cdot ds}{d\theta} + f(s) = F(\theta)$$

I have not been able to discover a general solution of this differential equation, which can be expressed in terms of ordinary functions. It can, however, be developed by the short-arc method shown in Par. 55, the values of $F(\theta)$ being known from the characteristics of the driven machine, and the values of $f(s)$ being read from the torque-speed chart of the proposed motor. I have plotted a large number of curves for different motors and machines in this way, one of which is shown in Fig. 3.

![Torque-Speed Chart of Motor](image-url)