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STRESS DISTRIBUTION IN ELECTRIC-RAILWAY MOTOR PINIONS AS DETERMINED BY THE PHOTO-ELASTIC METHOD

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This paper embodies some results of a general scientific study undertaken by the General Electric Company for the development of superior electric-railway motor pinions. The particular portion of the work described was performed at the Massachusetts Institute of Technology, using the General Electric Company's apparatus for stress determination in transparent models by the photo-elastic method. Some of the supplementary mechanical tests were made at Schenectady, and throughout the work close contact was maintained with the Railway Motor Department and the Research Laboratory at Schenectady. A brief description and discussion of the photo-elastic method is given in the first part. The stress distribution in, and the causes of ruptures of, given types of gear pinions used in electric-railway motors, as investigated by the photo-elastic method, are afterward reported upon and discussed.

I — DESCRIPTION OF THE METHOD

HOW TO DEFINE THE STATE OF STRESS AT ANY POINT OF A SOLID BODY

THE state of stress at any point in a solid body is determined when the traction across every plane through the point is known. There exist at any point three orthogonal planes across which the traction is purely normal and which are called the planes of principal stress. The normal tractions across those planes are called the principal stresses. The state of stress at any point is completely determined by the direction and the magnitude of the principal stresses at the point under consideration. The principal stresses, given in direction and in magnitude, express in the most

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general and complete way the elastic state at any given point. The bending moment, the shearing forces, etc., are readily deduced from the direction and the magnitude of the principal stresses. Furthermore, one of the principal stresses always expresses the maximum stress.

2 The notion of principal stress may be illustrated as follows:

3 Consider a spherical element in a solid body. External applied loads will deform this spherical element into an ellipsoidal element (Fig. 1). The axes of this ellipsoid will correspond in direc-

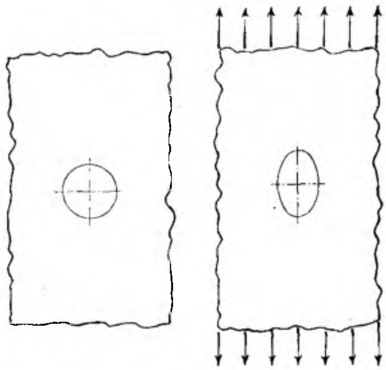


FIG. 1 ELLIPSOIDAL ELEMENT RESULTING FROM SUBJECTING A SPHERICAL ELEMENT TO STRESS

tion and in magnitude to the direction and the magnitude of the principal stresses.

4 The orientation and the form of the ellipsoid, and therefore the direction and the magnitude of the principal stresses, will define the state of stress at the point under consideration.

5 The axes of the ellipsoid represent the largest and the smallest deformation at the point under examination. Correspondingly, the principal stresses give the direction and the magnitude of the maximum and the minimum stress.

6 If the three principal stresses vary from point to point in the structure, the problem to be dealt with is a three-dimensional elastic one. If one of the three principal stresses vanishes throughout, it is a two-dimensional elastic or plane-stress problem.

7 Corresponding to the three- and two-dimensional elastic-stress problems there are also the three- and two-dimensional elas-

tic-strain problems, when the deformations corresponding to the principal stresses are considered.[1]

8 A great number of structural problems (bridge, ship, airplane, plate, dam, etc., construction) are, or their stress analysis may be reduced to, two-dimensional elastic problems.

THE PHOTO-ELASTIC METHOD OF STRESS DETERMINATION

9 As set forth in Par. 1, the state of stress at any point is most completely defined by the direction and the magnitude of the principal stresses. These are, therefore, the elements which we wish to determine for a complete analysis.

10 The photo-elastic method solves the two-dimensional elastic problems. It primarily takes advantage of the double refracting properties shown by isotropic transparent substances when put under stress. The stresses in the structure may therefore be determined from models made of a homogeneous transparent material, and ordinarily on a reduced scale. The stresses in a steel, cement, or any other structure, homogeneous throughout and obeying Hooke's law of linear proportionality between stress and strain, may be readily deduced from the values obtained by the analysis of the corresponding transparent model for the case of two-dimensional elastic problems.

11 If plane polarized light is passed through a stressed specimen of celluloid and afterward through a second nicol prism whose principal section is parallel to the plane of polarization of the original beam of light, only the points where the principal stresses are respectively parallel and perpendicular to the principal sections of the crossed nicols remain dark. This result makes it possible to determine the directions of the principal stresses at any given point. Moreover, this information is needed for the measurements which will be described later.

12 If now circularly polarized light be passed through the specimen, by interference of the two component rays, which in the double-refracting specimen have suffered a relative retardation at each point proportional to the difference in magnitude of the two principal stresses, a colored image is obtained. (Figs. 2, 3, 4 and 5.)

13 By a comparison method, based upon the interposition in

[1] A complete theory of stress and strain may be found in the Treatise on the Mathematical Theory of Elasticity by A. E. H. Love, 3rd ed., chapters i–iv.

the proper direction of a comparison member of constant cross-section, put under uniform tension in a suitable frame (Fig. 6), the value of the difference of the principal stresses at any given point may be read on the dynamometer of the frame.

14 Now, in the two-dimensional elastic problems the transverse deformation, i.e., the deformation along a normal to the plane of the two principal stresses, is proportional to the sum of those two stresses. By means of a lateral extensometer, Fig. 7, we measure this transverse deformation.

15 From the values of the differences and the sums of the principal stresses, the separate values of each of them are computed, thus determining completely the state of stress.

16 A question naturally arising is whether the results obtained on a transparent body such as celluloid hold for structural materials.

17 It is shown by the general discussion of the equations of elastic equilibrium that in the case of strain or plane stress in an isotropic body obeying Hooke's law of linear proportionality between stress and strain, the stress distribution is independent of the moduli of elasticity and consequently of the material of which the body is made. Thus the stress distribution experimentally determined in the case of a celluloid body is the same as it is when the body is made of any other isotropic substance such as iron, steel, etc., obeying Hooke's law, in distribution, direction, and magnitude.[1] Moreover these conclusions derived from the general theory of elasticity have been checked by experiment.[2]

18 The photo-elastic method can be applied to the great majority of structural problems, not only in taking the place of mathematical computation, but particularly in solving those struc-

[1] Except, however, if the body is multiply connected and the resultant applied forces do not vanish separately over each boundary. In this particular case the correction coefficients for passing from one isotropic substance to another may be experimentally determined. (On Stresses in Multiply-Connected Plates, by L. N. G. Filon, British Assn. Report, 1921.)

[2] Photo-Elastic Measurements of the Stress Distribution in Tension Members Used in the Testing of Materials, by E. G. Coker, Excerpt Proc. Inst. C. E. (London), — vol. ccvii, part ii, p. 8.

Photo-Elastic and Strain Measurements of the Effects of Circular Holes on the Distribution of Stress in Tension Members, by E. G. Coker, Trans. Inst. Engrs. & Shipbuilders in Scotland, vol. lxiii, part i, p. 33.

La Photo-Elasticimétrie, ses principes, ses méthodes et ses applications, by Paul Heymans. Bull. Soc. Belge Ing. et Ind., Aug. 1921, pp. 147–154, 165–167, 189–199.

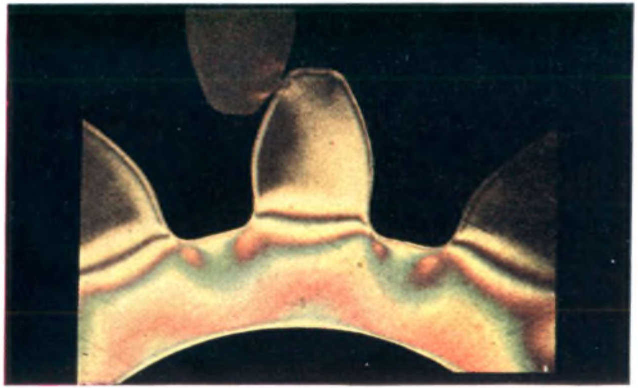


Fig. 2. Colored Image when Normal Inside Pressure Alone is Applied

Fig. 3. Colored Image when Both Normal Inside Pressure and Maximum Torque are Applied

Fig. 5. Colored Image Obtained for Normal Inside Pressure and Reduced Torque

Fig. 4. Colored Image when Both Normal Inside Pressure and Maximum Torque are Applied

tural problems where mathematics becomes too involved to be of help. Moreover it has the great advantage of giving the maximum stress at each point throughout the whole structure, and it there-

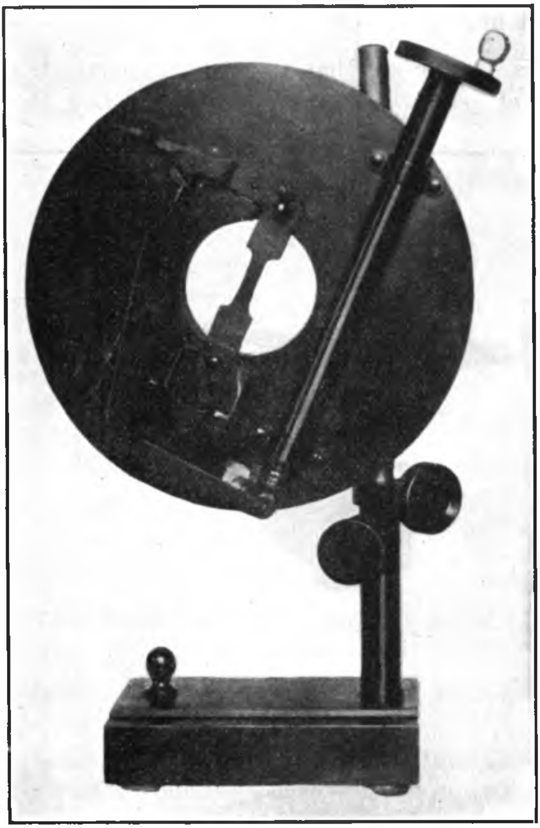


FIG. 6 FRAME FOR COMPARISON MEMBER DESIGNED BY E. G. COKER AND A. L. KIMBALL, JR.

fore offers an effective means of increasing safety and reducing superfluous material.

II—A STUDY OF THE STRESS DISTRIBUTION IN GEAR PINIONS

19 When accidents occur with gear wheels, besides the metallurgical question, three possible causes of failure suggest themselves:

a The gear wheel may not have been properly designed
b It may have failed under an excessive load
c When the pinion was shrunk hot or forced on to a tapered shaft, an excessive inside radial pressure may have been set up.

20 It is easy to see that the ordinary methods of resistance calculations of gear wheels, based on considering the tooth as a

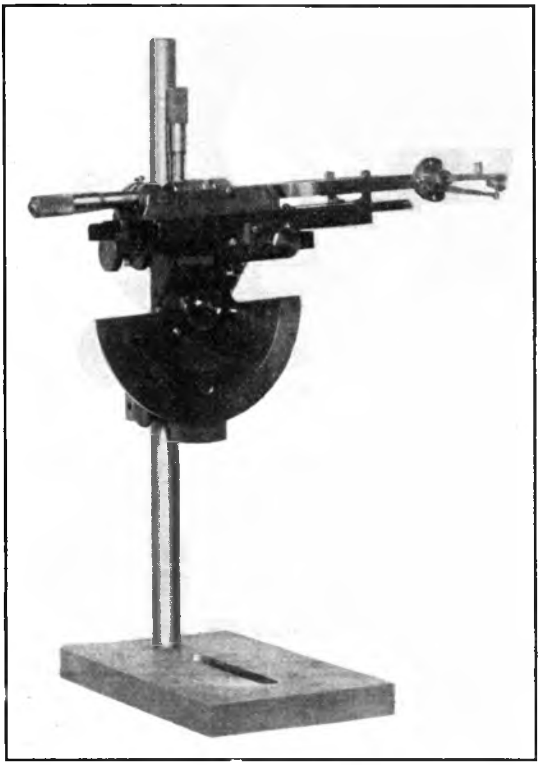


FIG. 7 LATERAL EXTENSOMETER DESIGNED BY P. HEYMANS

cantilever loaded at its end, would not be expected to give reliable and complete information as to stress distribution, not even for the root section of the tooth which is under consideration.

21 Indeed, the shape of the tooth, the curvature at the root, the ratio of the diameter of the pinion bore to the root and outside diameter, the permanent stresses introduced by the placing of the

pinion on the shaft, etc., all affect the stress distribution and the maximum stress. Photo-elastic analysis shows that these factors affect the stresses considerably more than would be expected from present methods of estimating. For standardized pinions the correction coefficients can only partially take account of these factors. For special pinions or for pinions of which more efficient running is required, a photo-elastic analysis seems to be the best if not the only effective way to determine the stress distribution and to locate the maximum stress.

22 A detailed analysis of the stress distribution determined

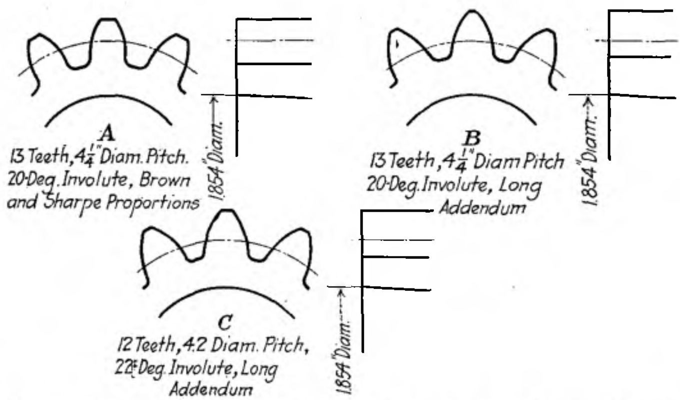


Fig. 8 Tooth Forms of Pinions Subjected to Photo-Elastic Analysis

for different gear pinions and under different loading conditions is given below.

23 The authors wish first to call attention to certain interesting points brought out by photo-elastic analysis, which have been checked by tests carried out on steel sections. These are particularly interesting because they are unexpected.

24 Besides the stress distribution in the different sections of the pinions represented by Fig. 8, the photo-elastic analysis has given as maximum stress under normal inside radial pressure and maximum torque:

80,000 lb. per sq. in. for tooth form *A*
70,350 lb. per sq. in. for tooth form *B*
60,900 lb. per sq. in. for the tooth *C*.

Moreover the 12-tooth pinion shows, besides a smaller maximum stress, a better stress distribution.

25 For steel pinions the maximum stress attained under normal conditions, although high, appears not to be excessive. *Tooth C appeared to be a better design* under normal conditions.

26 The stresses due to shrinking or forcing the pinion on the shaft can only be estimated. The pinion may be assumed to

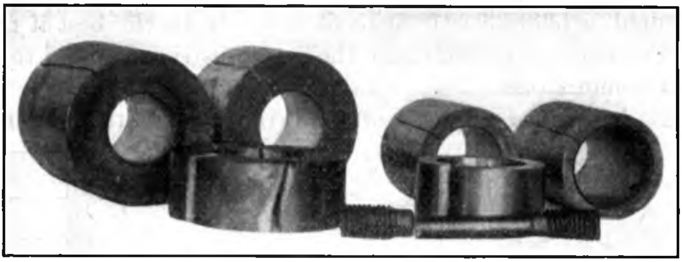


FIG. 9 STEEL RINGS RUPTURED BY BEING FORCED ON TO A TAPERED PLUG

be a plain circular ring, for which case the stresses may be mathematically computed. The stress at any point of the ring as well as the maximum stress in the ring depends upon the lengths of the inside and outside radii. The opinion generally expressed is that for the case of the pinion the maximum stress will be intermediate

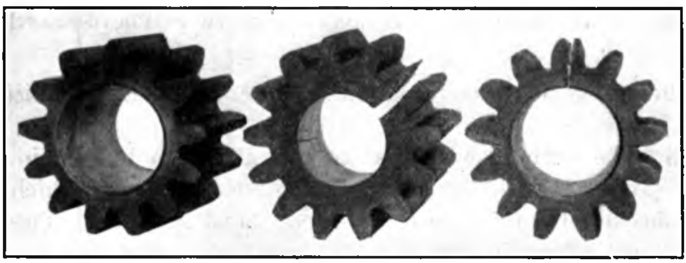


FIG. 10 STEEL PINIONS RUPTURED BY BEING FORCED ON TO A TAPERED PLUG

between the maximum values obtained for rings of which the outside diameters are respectively equal to the root diameter of the tooth and to the outside diameter of the pinion, the inside bore being the same.

27 Photo-elastic analysis shows that *the gear pinion is even weaker than the plain circular ring whose outside diameter is equal to the root diameter of the tooth.* The change of external profile, due to

the presence of the teeth, although requiring an addition of material, weakens the structure.

28 Figs. 9 and 10 show the steel specimens after having been tested by forcing a tapered plug into the bore; and Table 1 gives the rupture load applied to the tapered arbor forced into the bore for the different specimens. These confirm the photo-elastic results.

29 Previous to the photo-elastic investigation of the stresses due to radial inside pressure in pinion sections, fracture due to pure radial inside pressure would have been expected to occur through the minimum radial cross-section.

30 From Fig. 2, representing the color image obtained in the photo-elastic analysis, it appears that *the regions under the teeth are under higher stress* and that the points at the inside boundary right under the teeth are points of maximum stress.

TABLE 1 RUPTURE LOAD ON ARBOR FORCED INTO SPECIMENS TESTED

	Inside diam., in.	Outside diam., in.	Root diam., in.	Rupture load, lb.
Ring	1.854	3.5	.....	85,000
Ring	1.854	2.5	.....	51,000
Pinion	1.854	3.5	2.5	47,000

31 Fig. 10 gives the fractures obtained on steel sections. Two of the sections show fractures right through the thickest layer of material, while all of them started at points where the photo-elastic analysis had revealed maximum stress. The unevenness of the material must account for the deviation of the fracture in one of the cases.

32 Can any statement be made as to the causes of the failure by inspection of the shape of the fracture? In the case in which the authors were interested, the photo-elastic analysis determined the best design. As before said, either the placing of the pinion on the shaft, if carelessly done, for instance by pounding the pinion heavily on the tapered shaft, or excessive torque and blows due to sudden meshing or the taking on of a heavy load, will set up dangerous stresses.

33 The authors' photo-elastic analysis has shown that *the sections of dangerous stresses are different for different values of inside radial pressure and applied torque load.*

34 The fracture shown in Fig. 11 is of an *open* V-shape. Photo-elastic analysis shows that *the higher the inside radial pressure becomes,* for a given torque load, *the sharper becomes the V-shape of*

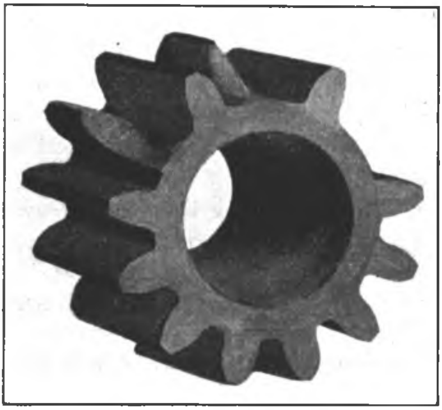


FIG. 11 FATIGUE FAILURES OF TEETH PRODUCED BY EXPERIMENT (WITHOUT RADIAL PRESSURE IN BORE)

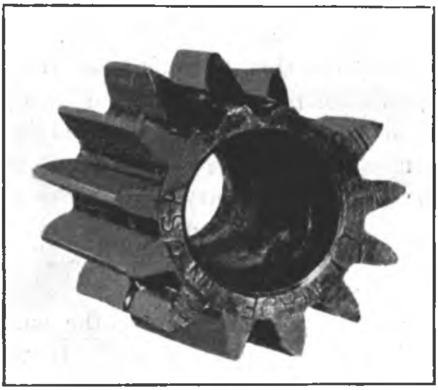


FIG. 12 FATIGUE FAILURES OF TEETH PRODUCED BY EXPERIMENT (WITH HEAVY RADIAL PRESSURE IN BORE)

the section of dangerous stresses. (Fig. 12.) If the fracture is due to too high a torque load, the angle of the V will approach 180 deg. Tests on steel sections have been made with a specially built impact machine.

35 Without inside radial pressure the fracture obtained is a

straight line through the root section of the tooth. With increasing pressures the V-shaped fracture becomes sharper. For an inside radial pressure exceeding the elastic limit, however, the observation does not hold. The reason for this departure from what the photo-elastic method had predicted is to be found in the fact that beyond the elastic limit the stress-and-strain relation no longer follows Hooke's law. Therefore the stresses set up in the steel pinions by the shrinking process no longer correspond with those set up in the celluloid model.

36 While the flat shape of the break in Fig. 11 is one limiting case (torque without radial shrinking pressure), Fig. 10 may be considered as the other limiting case (radial shrinking pressure without torque), showing a V-shaped fracture for which the angle of the V has become equal to zero.

37 It may be concluded, then, that the inspection of the fracture may be a means of determining the cause of the failure. In this way, possibly, the responsibility may be established between builder and customer as regards pinion mounting.

THE DETAILED STRESS ANALYSIS

38 *External Forces Applied to the Pinion When in Service.* The pinion is shrunk on to the shaft after having been bored so as to fit the shaft at a temperature of 160 deg. fahr. above normal room temperature.

39 In normal working conditions, the torque load to which the pinion is subjected corresponds to a tractive force of 500 lb. per inch of face of the tooth, tangent in direction to the pitch circle. The whole torque is supposed to be transmitted by a single contact.

40 Calling respectively $\widehat{rr}$ and $\widehat{\theta\theta}$ the radial and the tangential principal stress in a circular ring, of which the outside diameter equals the root diameter of the teeth, the inside bore being the same as the pinion bore, $(\widehat{rr} - \widehat{\theta\theta}) = 28{,}800$ lb. per sq. in. for $\Delta t = 160$ deg. fahr. This value of $(\widehat{rr} - \widehat{\theta\theta})$ is the stress value of the color bands obtained in polarized light (isochromatic bands), and will therefore be used in the stress analysis of the celluloid model to secure the right expansion pressure before the torque is applied. For radial pressures higher than this normal shrinking pressure, the same characteristic of the $(\widehat{rr} - \widehat{\theta\theta})$ value will be used.

41 The tangential tractive force is applied at varying dis-

tances from the root of the tooth, depending upon the point of contact. The most unfavorable conditions arise when this force is applied at the top of a single tooth. Moreover the starting torque load being higher than that realized under normal running conditions, the applied tractive force was brought up from 500 lb. to 1500 lb. per inch of face.

42 Let us for convenience call:

a The *normal* inside pressure, the value of 28,800 lb. per sq. in. for $(\widehat{rr} - \widehat{\theta\theta})$, corresponding to a shrinking pressure due to a temperature variation of 160 deg. fahr.

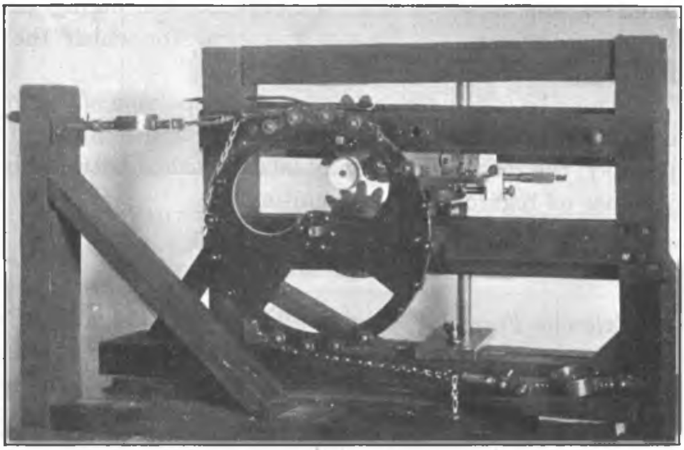


Fig. 13 Frame Used for Applying Loads to Celluloid Models of Pinions

b The *maximum* torque, the torque corresponding to a tractive load F of 1500 lb. applied normally to the contour of the tooth (condition of contact) at the top of one pinion tooth

c The *normal* torque, the torque corresponding to a tractive load F of 500 lb. applied under the same conditions as above

d *Increased* inside pressures, the values of $(\widehat{rr} - \widehat{\theta\theta})$ exceeding the normal inside pressure, as defined above.

43 *The Photo-Elastic Analysis.* Fig. 13 represents the frame used for the loading of the models. A tapered expansion ring is

used to produce the radial inside pressure. The torque is measured by properly mounted dynamometers.

44 The first sets of measurements were made under normal

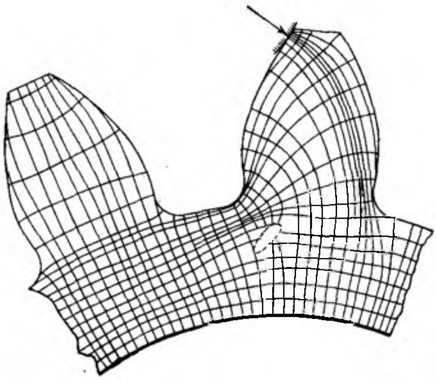


FIG. 14 LINES OF PRINCIPAL STRESS DETERMINED BY POLARIZED LIGHT—NORMAL INSIDE PRESSURE AND MAXIMUM TORQUE LOAD

inside pressure and maximum torque load Fig. 14 represents the lines of principal stress, deduced from the isoclinic bands. The

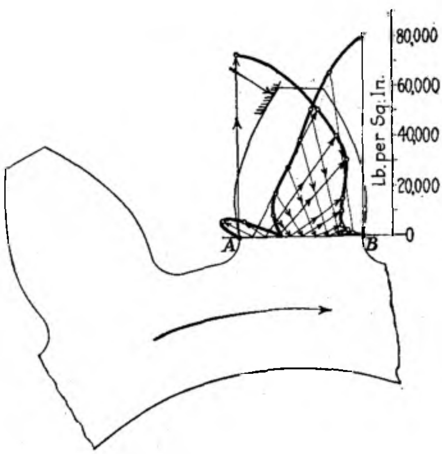


FIG. 15 CURVES SHOWING THE TWO PRINCIPAL STRESSES IN DIRECTION AND MAGNITUDE FOR POINTS ALONG THE SECTION *AB*

tangents to these lines represent at each point the directions of the principal stresses.

45 Fig. 2 gives the colored image when the normal inside pressure alone is applied, whereas Figs. 3 and 4 give the image

obtained when both the normal inside pressure and the maximum torque are applied. An optical measurement on the image shown in Fig. 2 allows one to adjust properly the amount of inside pressure before the torque is applied.

46 The determination of the values of the difference $(p - q)$ of the principal stresses is made on the image shown in Fig. 4. One of the two principal stresses vanishes at a boundary where no external forces are applied. In this case the optical measurements of the values of $(p - g)$ give directly the values of the tangential stress.

47 Inside of the body the optical measurements are supplemented by measuring the transverse change of thickness, which gives the values of the sum $(p + q)$ of the principal stresses.

TABLE 2 VALUES OF THE PRINCIPAL STRESSES ACROSS THE MINIMUM CROSS-SECTION OF THE LOADED TOOTH

Tenths of distance *AB* (Fig. 15) measured from *A*	*p* lb. per sq. in.	*q* lb. per sq. in.
0	0	72,600
0.1	13,850	57,300
0.2	10,450	49,000
0.3	3,710	41,700
0.4	−10,620	25,800
0.5	−20,300	18,700
0.6	−29,000	11,900
0.7	−40,000	9,000
0.8	−51,900	......
0.9	−65,700	5,320
B	−80,000	0

48 From the values of the principal stresses at a given point it is easy to obtain the stress on a section in any given direction at that point. Moreover, as said before, the two principal stresses represent respectively the maximum and the minimum stress. Thus the larger of the principal stresses will always give at each point the maximum stress in direction and magnitude.

49 At the edges where one of the principal stresses has vanished the values of $(p - q)$ and $(p + q)$ must correspond, i.e., the optical determination of $(p - q)$ and the determination of $(p + q)$ must check.

50 Also as we know the total force acting normally to a given section, the graphical integral of the curve, obtained by plotting

the resultant stresses acting normally to this section, must correspond to the total force. In the case of the pinions the data for such a check are not available, except for the section *AB*.

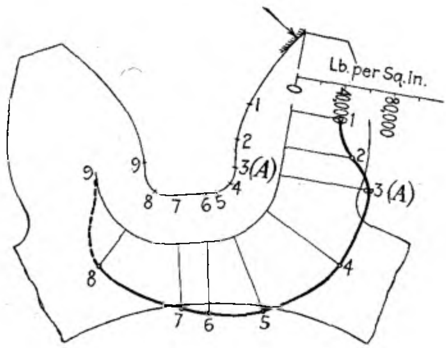


FIG. 16 TANGENTIAL STRESS AT TENSION SIDE — NORMAL INSIDE PRESSURE AND MAXIMUM TORQUE

51 Table 2 gives the values of the principal stresses through the minimum cross-section of the pinion tooth, to which the load is applied. The results given in this table have been plotted in

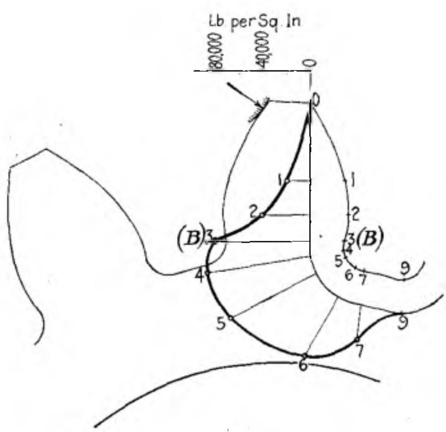


FIG. 17 TANGENTIAL STRESS AT COMPRESSION SIDE — NORMAL INSIDE PRESSURE AND MAXIMUM TORQUE

Fig. 15. At each point where measurements have been made the two principal stresses have been plotted in direction and in magnitude, the arrows serving to distinguish between tension and com-

pression. At the points A and B, $(p - q)$ and $(p + q)$ must check: they differ for A by 0.9 per cent and for B by 0.8 per cent.

52 The maximum tension occurs at A and is equal to 72,600 lb. per sq. in. The maximum compression occurs at B and is equal

TABLE 3 VALUES OF THE TANGENTIAL STRESS AT THE BOUNDARY OF THE LOADED TOOTH — TENSION SIDE

No. of point on Fig. 16	q lb. per sq. in	
1	41,000	
2	54,100	
3(A)	72,300	72,750 [1]
4	73,200	
5	64,800	
6	57,600	
7	54,100	
8	41,000	
9	......	

[1] Value obtained by taking ½ $[(p + q) + (p - q)]$, the other values being $(p - q)$ measurements.

to 80,000 lb. per sq. in. This difference between the absolute values of these stresses is of course due to the pressure on the inside of the pinion, which affects the tension and the compression stresses differently.

TABLE 4 VALUES OF THE TANGENTIAL STRESS AT THE BOUNDARY OF THE LOADED TOOTH — COMPRESSION SIDE

No. of point on Fig. 17	p lb. per sq. in.	
1	20,500	
2	41,000	
3(B)	79,500	80,000 [1]
4	80,000	
5	82,200	
6	60,000	
7	29,000	
8	0	

[1] Value obtained by taking ½ $[(p + q) + (p - q)]$, the other values being $(p - q)$ measurements.

53 Figs. 16 and 17 give the values of the tangential stresses along the edge of the tooth on which the load is applied. The numerical results of Table 3 have been plotted in Fig. 16, this table giving the tangential stresses at the tension side. Also the numerical

results of Table 4 have been plotted in Fig. 17, this table giving these stresses on the compression side. Since no external load is applied at this side, the optical measurements give the values of the tangential stresses up to the top of the tooth.

54 Table 5 and Fig. 18 give the numerical and plotted values of the stress difference $(\widehat{rr} - \widehat{\theta\theta})$ along the inside boundary of the pinion, the normal inside pressure and the torque load being applied. A circular ring to which a uniform inside pressure is applied will show concentric isochromatic bands. The deflections of those bands (Fig. 2) in the case of the pinion show the disturbance due to the presence of the teeth.

55 When the maximum torque is applied, the values obtained for $(\widehat{rr} - \widehat{\theta\theta})$ give the curve of Fig. 18. The colored images

TABLE 5 VALUES OF $(\widehat{rr} - \widehat{\theta\theta})$ ALONG THE BOUNDARY OF THE BORE

No. of point on Fig. 18	$(\widehat{rr} - \widehat{\theta\theta})$ lb. per sq. in.
1	36,600
2	54,100
3	36,600
4	18,100
5	41,000
6	61,500
7	43,500
8	38,700

as well as the diagrams show that the load applied at the top of one tooth extends its influence as far as the inside boundary of the pinion. The combination of the inside uniform pressure, already disturbed by an irregular outside boundary, with irregularly distributed stresses — tensions in certain parts and compressions in others — due to the torque load, do not of course give a resultant stress distribution which shows any symmetry with respect to the point of contact. The upper pinion being the driving pinion, it may be seen on the colored image (Fig. 3) that the stresses vanish rather rapidly in the withdrawing part, but that the penetration extends much farther into the approaching part.

56 It may also be interesting to point out that there is a zone of zero stress inside of the pinion under the root of the tooth when the torque load is applied. This is often the case at points where lines of principal stresses converge.

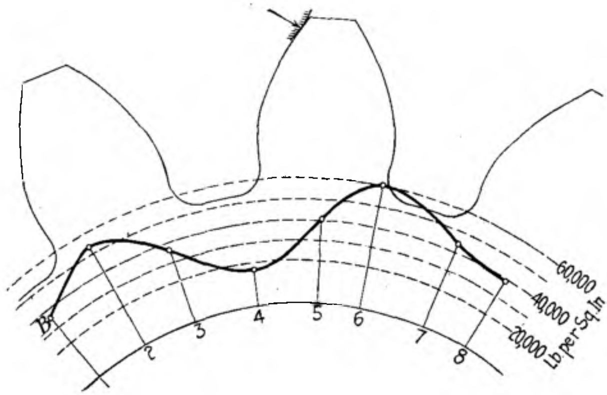


FIG. 18 STRESSES ALONG THE INSIDE OF BORE, WITH NORMAL PRESSURE (28,820 LB.) AND MAXIMUM TORQUE

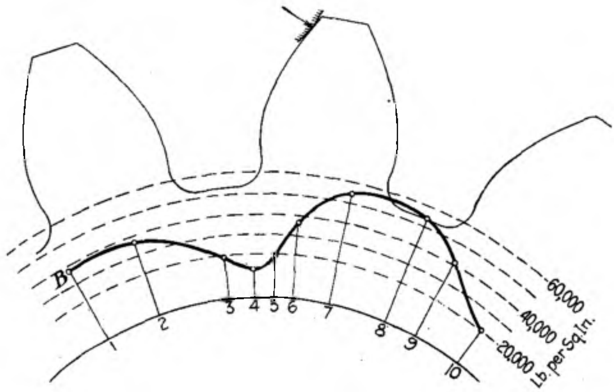


FIG. 19 STRESSES ALONG THE INSIDE OF BORE WITH DECREASED PRESSURE (18,100 LB.) AND MAXIMUM TORQUE

57 The question of engineering interest was to find the relative influence of the factors which affect the maximum stress, and the authors therefore varied the values of:

a The inside normal pressure
b The torque load.

58 The values of $(\widehat{rr} - \widehat{\theta\theta})$ along the inside boundary when the maximum torque load is applied are given in Table 6 and have been plotted in Fig. 19 for the case of reduced inside pressure. The colored image did not show noticeable variation across the mini-

mum cross-section AB and along the outside edges of the main tooth. The influence of the inside pressure within the above-mentioned limit does not affect materially the regions of maximum stress, due in this case to the torque load.

TABLE 6 VALUES OF $(\widehat{rr} - \widehat{\theta\theta})$ ALONG THE BOUNDARY OF THE BORE

(Maximum torque — reduced radial pressure)

No. of point on Fig. 19	$(\widehat{rr} - \widehat{\theta\theta})$ lb. per sq. in.
1	37,600
2	36,600
3	20,500
4	14,550
5	20,500
6	36,600
7	54,100
8	54,100
9	41,000
10	20,500

59 Fig. 5 shows the image obtained for normal pressure and reduced torque. Having applied 0.7 of the maximum torque value, the stresses showed a general reduction in the region of high stress.

TABLE 7 VALUES OF THE TANGENTIAL STRESS ALONG THE BOUNDARY OF THE LOADED TOOTH — TENSION SIDE

(Normal inside pressure — reduced torque)

No. of point on Fig. 20	q lb. per sq. in.
1	39,700
2	51,500
3	58,500
4	57,700
5	56,600
6	51,500
7	38,000
8	19,500

The values of the tangential stresses along the tension side of the boundary of the main tooth are given in Table 7 and are plotted in Fig. 20. This should be compared with the same diagram (Fig. 16) for the case where the full load is applied. The maximum tension has dropped from 73,200 lb. per sq. in. (Table 2) to 57,700 lb. per sq. in. (Table 7); i.e., it has been reduced to 0.8 of its previous value. The fact that it has only dropped to 0.8 whereas the torque was reduced to 0.7, is explained by the permanent stress due to the inside radial pressure which had been maintained at its previous value. A reduction of the torque load has as a result a reduction of the maximum stress. We shall see later that this is not always the case.

60 When the inside radial pressure is increased in such proportion that without any torque being applied it produces stresses at the outside boundary of the gears of a magnitude approaching

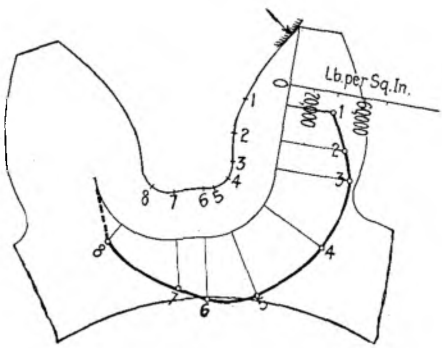


FIG. 20 TANGENTIAL STRESS AT TENSION SIDE — NORMAL INSIDE PRESSURE AND REDUCED TORQUE

that due to the torque load, it will be this internal pressure which will have a preponderant influence.

61 Pinions have been examined with maximum values for $(\widehat{rr} - \widehat{\theta\theta})$ of 60,000 and 81,500 lb. per sq. in. at the inside boundary with the torque load at its normal value of 500 lb. tractive force per inch of face. The tractive force was afterward brought up to its maximum value of 1500 lb.

62 These tests showed that the torque load, when applied to the pinion subjected to those increased radial pressures, affects only the distribution of the stresses. It makes the high stresses extend over a larger area, but it does not increase materially the maximum stress. In these cases the dangerous section is no longer a straight section through the root of the tooth but it follows a V-shaped line, the lower point of which lies toward the inside bore. The sharpness of the angle of the V-shaped fracture at the base of the tooth appears to be due to an excess of radial shrinking pressure.

In practice this excess is due to the pounding of the pinion on to the tapered shaft past its normal position.

63 In this connection a study was made of the stress distri-

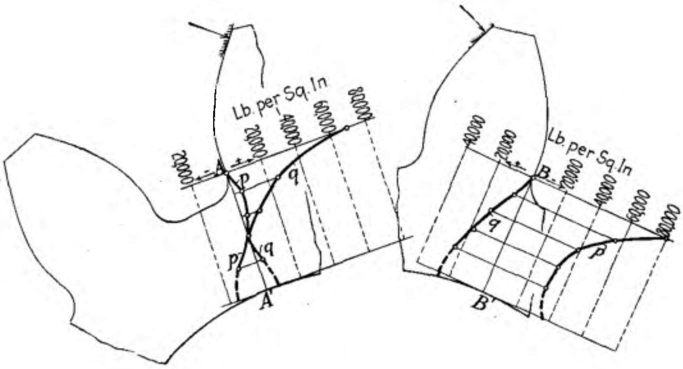


FIG. 21 PRINCIPAL STRESSES ACROSS RADIAL SECTIONS OF TOOTH — NORMAL INSIDE PRESSURE AND MAXIMUM TORQUE

bution through two radial sections, passing respectively through the points A and B of the minimum cross-section of the main tooth, the points of maximum tension and compression.

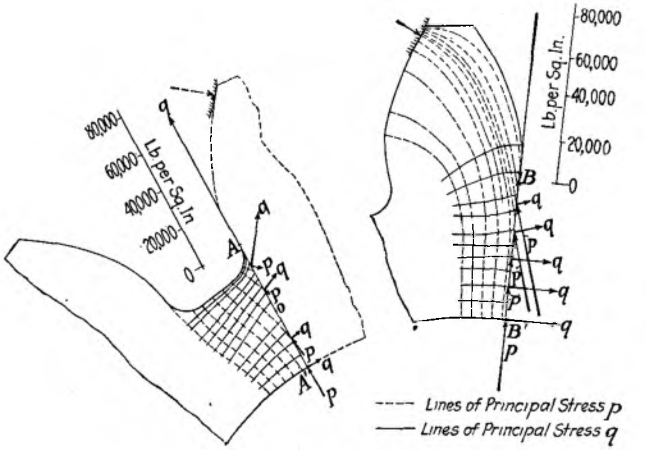


FIG. 22 PRINCIPAL STRESSES IN DIRECTION AND MAGNITUDE FOR SAME RADIAL SECTIONS AS THOSE SHOWN IN FIG. 21

64 The values of $(p - q)$ were deduced from the colored image of Fig. 4. Extensometer measurements of $(p + q)$ were made. As before, the scales of both measurements were determined

so that the stresses in the models should represent the stresses in the steel pinion.

65 The maximum torque and the normal inside pressure were applied. Table 8 and Figs. 21 and 22 give the values obtained. Fig. 21 gives the magnitude of the principal stresses along the two sections AA' and BB'. Fig. 22 gives a portion of the lines of principal stress taken from Fig. 14, and for the same sections AA' and BB' shows the two principal stresses plotted in direction and in magnitude.

TABLE 8 VALUES OF THE PRINCIPAL STRESSES ACROSS THE RADIAL SECTIONS PASSING RESPECTIVELY THROUGH THE POINTS A AND B OF THE MINIMUM CROSS-SECTION OF THE LOADED TOOTH

Cross-Section BB', Fig. 21; Distance in inches from point B'	p lb. per sq. in.	q lb. per sq. in.
0.410 (B)	−79,900	0
0.334	−55,800	6,000
0.256	−39,000	15,200
0.179	−32,600	18,400
0.102	−27,100	23,500
Cross-Section AA', Fig. 21; Distance in inches from point A'		
0.410 (A)	0	69,350
0.334	4,350	25,400
0.256	2,700	10,300
0.179	0	0
0.102	−11,350	3,350

66 A good way to visualize the state of stress at a given point is to consider a rectangular element with its sides parallel to the two principal stress directions at that point. By considering such elements along the sections AA' and BB' (Fig. 22) from this viewpoint, one can form a mental picture of how the section is acted upon by the elastic forces.

67 It would require too much space to include in this paper a full discussion and to make a complete report of the results summarized here. The authors trust that the material they have presented will stimulate those interested in this subject to further efforts in the development and use of the photo-elastic method.

68 It seems, finally, almost superfluous to call attention to the comparative ease with which such a stress problem as this can be handled by the photo-elastic method, whereas the use of ordinary engineering methods gives untrustworthy results and the exact mathematical solution based upon the theory of elasticity is impossible.

69 Acknowledgment is due to the Massachusetts Institute of Technology for permission to use in this article certain of the results included in the thesis entitled Photo-Elasticity or Stress Analysis by Means of Optical Methods [1] submitted by Dr. Paul Heymans, University of Ghent, Belgium, as partial fulfillment of the requirements for the degree of Doctor of Science from the Institute.

DISCUSSION

S. Timoshenko. The photo-elastic method used in the experiments described in this paper is by no means new, having found practical application in many technical problems in the last twenty years. It was employed by Mesnager [2] in the stress distribution in elastic arches, by E. G. Coker in the bending of a beam subjected to the action of a concentrated force,[3] in the effects of circular holes on the distribution of stresses, in the stress distribution in tension members used in the testing of materials, in contact pressures, and quite recently in conjunction with K. C. Chakko [4] in the study of the action of cutting tools.

The photo-elastic method of analysis of two dimensional problems of elasticity has a broad application within certain narrow limits and it is important that these limitations be clearly understood in order to use the method correctly and to the best possible advantage. This method of analysis fails where the elastic limit of the material under test is exceeded. In view of this limitation, however, a marked simplification of the work can be made. Because of the proportionality between stress and strain within these limits, the principle of super-position can be used. According to

[1] Abstract in *Tech. Eng'g. News,* May 1922.

[2] *Annales des Ponts et Chaussées,* 1901, 1913.

[3] Edinburgh Roy. Soc. *Trans.,* vol. 41, 1904.

[4] *Proc.,* Institution of Mechanical Engineers, May 1922.

this principle, each force produces its effect independently of any other force and the resultant effect of several simultaneous forces is the vector sum of the individual effects produced by each of them alone. If two observations had been made by the authors — i.e., (1) the stress distribution for some external force acting on the tooth of the pinion and (2) the stress distribution for some pressure acting on the inner diameter — the stress at any point of the pinion, due to any combination of force and pressure (provided that the resultant stress lies within the elastic limit), can be obtained by the above mentioned principle.

It is easy to see how in this manner we have all the information obtainable, instead of just a few isolated facts, and generalizations can be made accordingly. This gives a method of study. It is a simple enough matter to set up the apparatus for getting the stress distribution for a pressure acting on the inner boundary. In the case of the force, however, we have the problem of holding the pinion fixed without materially affecting the stress distribution in which we are actually interested. One way to accomplish this object is to make the pinion and shaft [1] out of one piece of celluloid and to provide the necessary reaction to the force by clamping of the shaft. Another manner of procedure is the following: First, get the stress distribution for the simultaneous action of a given external force and inner pressure, then remove the force so as to obtain the stress distribution due to the inner pressure alone. Evidently, for any cross-section desired, the stress due to the force is the difference between the stresses obtained from experiment (2) and (1).

I am doubtful about the authors' results obtained for stresses along the inner boundary, because of the probability of slight irregularities in the shape of the surface which would produce inequalities of pressure and because of the great probability of initial strains produced by the machining of the sheet of celluloid. It may be well not to attach too much value to the results of these tests, since the presence of a shaft (which necessarily must be longer than the thickness of the pinion) will cause three dimensional stresses at the common boundary of pinion and shaft, for which the photo-elastic method is no longer valid.

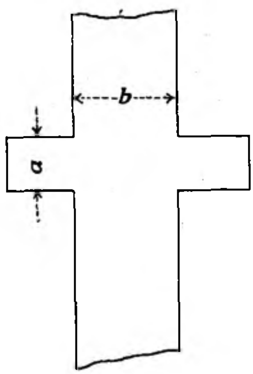
In Par. 58, the authors make the statement that the inside pressure does not materially affect the stresses along the outside

[1] The length of the shaft normal to the pinion will be equal to the thickness of the sheet of celluloid from which the model is cut.

edges of the main tooth. Then, certainly, the stress values for the points 1 and 2 in Fig. 20 should be 0.7 of those in Fig. 16 since the torque values are reduced in the same ratio and since it is the torque only which is responsible for these stresses. But the results of the paper do not confirm this fact. The values for these points, given in Tables 3 and 7, are:

	Table 3	Table 7	Ratio
Point 1	41,000 lb. per sq. in.	39,700 lb. per sq. in.	0.967
Point 2	54,100 lb. per sq. in.	51,500 lb. per sq. in.	0.955

In the study of the effect of pressure acting on the inner diameter, it would have been valuable to compare the pinion with a ring, of which the outer diameter is equal to the diameter at the root of the teeth. We do not expect to find a noticeable difference in the stress on the inner boundaries of these objects, but along the outer boundaries there must be considerable differences due to the presence of the teeth in the case of the pinion. The pinion is comparable with a bar which has sudden changes of cross-section at various points.[1] The conditions at these points, when the bar is put in tension, are similar to those produced at the homologous point in the pinion when pressure is exerted against its inner diameter. We expect the teeth to cause local increases in the stress, but the form of the curve by which they merge into the body of the pinion is of little influence, as in the case of the bar just cited. These local stresses decrease rapidly with increasing distance from the root of the tooth and, as stated in Par. 58, do not affect ma-



[1] We suppose a less than b as in the experiment of E. G. Coker. (See *Proc.*, Mech. Eng. 1921, p. 433.)

terially the stresses across the section which the authors designate as the minimum cross-section. Therefore, I cannot agree with the conclusion of the authors that the difference between the numerical values of the maximum tension (72,600 lb. per sq. in.) and maximum compression (80,000 lb. per sq. in.) is due to the pressure on the inner boundary of the pinion. This difference is more evidently due to the direction of the external force.

The authors, their statement to the contrary notwithstanding, certainly have all the information necessary for checking their work and I do not understand why they neglected to do so. They have, for instance, for the case shown in Fig. 15, the values of the external force and also the values of the stress across the section *AB*, as deducted from their experiments. The resultant force acting on this cross-section can be directly calculated from the known stresses, and it must necessarily balance the external force on the tooth.

In a plastic material such as steel, it is impossible to foretell that a given section will prove weakest at the breaking point of the material, because this section contains the maximum stresses when the body is stressed within the elastic limits. Therefore, the photo-elastic results cannot confirm any rupture tests on specimens. Moreover, no conclusions can be reached from the tests themselves. In all, only three were made and it seems to me that on the basis of these alone, it is just as probable that the breaks through the thicker sections were caused by the initial stresses in the materials as that they were caused by the stresses produced by the applied forces.

The present paper would be of direct benefit to the designing engineer if, for any given tooth shape, it gave him a definite relation between the actual stresses across the " minimum cross-section " of the tooth and those obtained by calculation from the cantilever formula. For example, taking the depth of the tooth above the cross-section considered as equal to $\frac{1}{2}$ in., the thickness equal to $\frac{1}{3}$ in. and the load at the end equal to 1500 lb., the maximum stress obtained from the cantilever formula is

$$p_{max} = \frac{1500 \times \frac{1}{2} \times 6}{(\frac{1}{3})^2} = 40{,}500 \text{ lb. per sq. in.}$$

Comparing this with the stresses 72,600 and 80,000 given by the photo-elastic method (see Table 2 of the paper), we see that the increase of maximum stress due to the local stresses near the root of the tooth, over that given by the cantilever formula, is 79 and 98 per cent.

If the method of study suggested earlier in this discussion had been used, it would have been possible to put the results in such a form that the engineer would know that the maximum stress, produced by a force of P lb. on a given shaped tooth would be equal to KF lb. per sq. in., where F is the maximum stress given by the cantilever formula and K is a constant determined by the researches.

In conclusion, I must point out some errors, which do not, however, affect the principal results of the paper.

In Par. 6, the authors assert that "if one of the three principal stresses vanishes throughout, it is a two-dimensional elastic problem." This conclusion is not correct. If we take, for instance, the torsion of a bar, one of the principal stresses at every point is equal to zero, but the problem is not a two dimensional one.

In Par. 56, it is stated that "several lines of principal stress can only intersect where the principal stresses vanish." This is not true. If we take, for instance, a circular disc subjected to a plane strain symmetrical about the center, the radii of the disc will be lines of principal stress. These radii all intersect at the center of the disc, where the stress is not necessarily zero.

It seems to me that the maximum stresses of 80,000, 70,350, and 60,900 lb. per sq. in. correspond rather to normal radial pressure and *maximum* torque, than to normal radial pressure and *normal* torque, as it is stated in Par. 24.

J. O. Madison.[1] We have on the road today three different motors; in the subway, 200-hp. motors; on the elevated, approximately 125-hp. motors; and on the surface, motors ranging from 4 to 65 hp.

In large motors there is plenty of material between the bottom of tooth and the shaft, from an inch and up to an inch and thirteen-sixteenths; and there is very little trouble holding the pinions on and very little trouble with splitting of pinions or even the breaking of teeth. They are boiled in water and put on hot with one blow of a 10-pound sledge.

On the elevated we have smaller pinions, and the distance is a trifle under an inch from the bottom of the tooth to the shaft. These pinions are secured by boiling and by hammering them on with three blows of a 10-pound sledge, and we are finding con-

[1] Engineer Car Equipment, Interborough Rapid Transit Co., New York, N. Y.

siderable trouble with them; we are having quite a number of tooth breakages and occasionally pinions split through the hub material. On the surface car pinions, where there is the smallest amount of material under the tooth, the pinions are put on by heating and also by hammering them on with approximately 8 to 10 blows of the 10-pound sledge. With these there is very great breakage, but we cannot apparently get along without putting them on thus securely, because if we do not hammer them on they loosen up in service.

It would appear, therefore, that we need some very strong material with good elongation on motor pinions, particularly where there is a small amount of material under the tooth.

Alphonse A. Adler. We should know all the fundamental facts about the design of pinions. The models should be rotated, as in practice, and a moving picture obtained in order to be sure that maximum stress occurs only at the time the teeth get into action. Then we would be in a position to determine what assumptions must be made in the elastic theory in order to obtain a useful formula.

E. O. Waters. It seems to the writer that there would be quite a field for this photo-elastic method in connection with the relation between stresses and pressure distribution and wear of material. Take, for example, the case of gear and pinion contact. The interesting question is whether that method would show the same stress distribution for new gears as for gears that have been in service for some time. Taken in the case of a bearing supporting a shaft, if the bearing is new, the contact might be at a few points, whereas if it was a well-worn bearing, the pressure ought to be fairly uniformly distributed. In those cases is it possible to use this method with actual samples of the material in question, using very thin sections? It seems that the experiment would be more direct if the same material was used as in actual service.

Paul Heymans, in answer to many questions, brought out the following points regarding this paper:

The celluloid models were mounted on steel shafts.

No scale for the values of the colors expressed in stress magnitude can be given independently of the nature of the material nor of the thickness at the point under consideration.

Besides the nature of the material, the interference effects, covering the colored image, depend upon the thickness of the material traversed by the ray. For models of constant thickness, the same table of correspondence between interference effect and stress holds. If the model is of varying thickness, a table of correspondence in function of the thickness has to be used for the interpretation of the colored images. However, in the photo-elastic analysis such a scale would not generally be of satisfactory accuracy. A comparison member put under uniform tension has ordinarily to be used.

The light band running along the edge of certain of the colored pictures is due to a superficial shrinkage caused by evaporation at the surface of certain of the volatile components of the celluloid after the model has been machined. This "edge effect" does not usually cause much inconvenience. Mr. T. H. Frost of our laboratory has recently gone into the developing of methods to prevent this shrinkage, and has obtained satisfactory results for celluloid of known composition.

No key-ways have been put in the models. The models represented a section of the pinion at the larger end of the tapered bore. The key-way originates at the opposite end and dies off gradually to nothing at the end which was analyzed.

The photo-elastic method allows the investigation of the stress distribution in all models through which it is possible to pass the ray of polarized light in a direction parallel to one of the three principal stresses. This is, of course, always the case for a two-dimensional elastic stress problem. As has been observed, the greatest majority of problems of engineering interest are two dimensional elastic stress problems.

The stress analysis carried out for these pinions has been made on homogeneous material, and consequently holds for homogeneous steel. However, as has been pointed out, case hardening does not change the values of the elastic constants entering in the general stress-strain relations, and consequently does not alter the isotropy of the material on elastic point of view. Therefore, the stress analysis in isotropic celluloid holds.

The photo-elastic stress analysis is facilitated when the model presents two faces parallel to the plane of the principal stresses which have to be determined. This is, however, not indispensable.

We may mention that we have in progress the stress analysis of the crankshaft of an airplane engine which by its shape and the numerous lubricating conduits represents a piece of very irregular contour.

Mr. Waters' remarks can be answered in a general way by observing that the photo-elastic method implies the building of a model in transparent material, in which the stresses may be studied under any given loading conditions. Regarding the specific question of the relation between stresses and wear of material in general and in the case of gears in particular, there is certainly quite a field for the photo-elastic method. The wear of material in gears and shafts, altering the meshing conditions and the conditions of the bearings, as affecting the stress distribution, could be investigated simply by considering a certain number of specific cases. It is, so far, not possible to investigate directly the stresses in steel pinions. The method is based on the effect of the stressed regions on polarized light passing through the specimen. This implies an investigation on transparent models. Regarding the use of very thin steel sections, it must be observed that the interference effects, originating the colored images, increase, up to a certain limit, with the thickness. Extremely thin sections would, therefore, at least disturb considerably the accuracy of the photoelastic measurements.

In fact, numerous fields of application of the photo-elastic method to very fundamental structural problems are already absorbing our attention, however anxious we may be to consider more of the problems which are proposed to us. We are gradually extending our laboratory with the hope that we will be able to respond to more of the questions which are of a troublesome nature to engineers.

THE AUTHORS. S. Timoshenko states that the method fails when the elastic limit of the material under test is exceeded. Of course this is true. The whole of the classical theory of elasticity applies only to cases of stress distribution where Hooke's Law is obeyed. Par. 17 of our paper covers this point.

It is, of course, true that the same stress distribution can be obtained by combining the stress distribution due to radial pressure and that due to the torque load according to the law of superposition. The stress distribution due to a torque load alone would be difficult to determine, as stated. Probably the best way to obtain this is to subtract the stress distribution due to the radial pressure alone from that due to the combined radial pressure and torque load. We then have the basis for combining these stress systems in any way desired.

In regard to the presence of the shift disturbing the plane stress distribution so that photo-elastic method fails, we would say that it may disturb slightly, but the photo-elastic method itself is only accurate to within two or three per cent and the effect mentioned is, in all probability, much less than this and has therefore been ignored.

In answer to the suggestion that we compare the pinion with a ring, the outer diameter of which is equal to the diameter at the root of the tooth, we refer to Par. 26 and 27 of our paper where that comparison is described. The authors disagree with the statement "it is just as probable that the breaks through the thicker section were caused by the initial stresses, etc.," in reference to rupturing steel pinions by forcing tapered plugs in them. The facts found were that three tests showed failures each time starting directly under the pinion which is just the point shown by the photo-elastic method to have the greatest principal stress difference. It is, to say the least, a striking result. We agree with Dr. Timoshenko to the extent that further experiment is desirable.

The definition in Par. 6 is only a definition and correct as it stands, and not a "conclusion" as stated by Dr. Timoshenko. In confirmation of this we refer him to Love's Mathematical Theory of Elasticity, Third Edition, Par. 94.

It is true that where the principal stresses intersect the stress is not necessarily zero.