STRESSES IN CYLINDRICALLY SHAPED ROTORS OF UNIFORM DIAMETER

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The author develops a general theory and analyzes mathematically the stresses in cylindrical rotors of uniform diameter such as are commonly associated with different classes of electrical machinery. The derivation of the fundamental equations is based upon the method used by Stodola in analyzing stresses in rotating disks of steam turbines. The general equations are derived and applications to seven different types of cylindrical rotors are analyzed and the formulas for these cases derived.

A KNOWLEDGE of the magnitude of the rotational stresses due to centrifugal force is of prime importance to engineers who are concerned with either the design or operation of high-speed rotating machinery, such as steam turbines, turbo-alternators, high-frequency generators, gyroscopes, and various other classes of electrical machines. The problem of determining the magnitude of the stresses is of especial interest to the designing engineer who is responsible for fixing the physical dimensions and proportions of particular machines which operate at given speeds.

2 The general theory and mathematical analysis of the stresses in rotating disks, as applied to steam turbines, is admirably treated in Stodola's classical treatise, The Steam Turbine. This analysis, however, to the author's knowledge, has never been extended to cover the types of rotors which are commonly associated with the different classes of electrical machinery. In fact, mechanical handbooks are practically the only available source of information on this subject and the data contained therein are usually very brief and applicable only to special cases. There has been a long-felt need by

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electrical engineers for a more complete and comprehensive analysis of the stresses in the rotating elements of electrical machines than is usually given in handbooks. In view of this fact it is the purpose of the author in the present discussion to present the physical conception and analysis of the stresses in several of the most important cases of cylindrically shaped rotors of uniform diameter. The derivation of the fundamental equations is based on the method used by Stodola and is given only as a matter of reference so that the reader can follow the discussion without referring to any other work. In order to make the physical conception of the analysis of the centrifugal stress phenomenon as clear as possible, two special elementary cases will be discussed before considering the actual cases of cylindrical rotors.
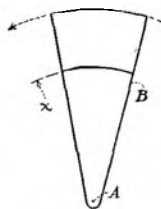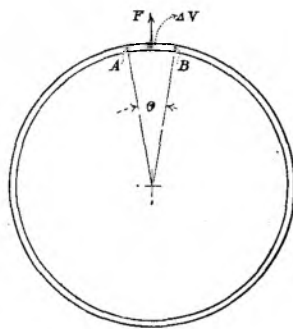

FIG. 1   FIG. 2

ANALYSIS OF THE STRESSES

3   If the rotating body consists of an isolated section B of a solid cylindrical rotor, Fig. 1, which rotates about an axis A at an angular velocity w, the total centrifugal force of the body acts in a radial direction and must be supported by the axis at A. At any distance x from the axis the centrifugal force of the external portion must be carried by the section of the material at this point. Hence the centrifugal force produces a radial stress in the material which varies in value from zero at the external radius to a maximum at the axis. The radial stress in the material causes it to elongate or stretch in the radial direction. Since the elongation is proportional to the stress, the radial elongation per unit length will be a maximum at the axis and at other points it will be proportional to the corresponding stress at these points.

4 On the other hand, if the isolated section consists of a thin annular cylinder as shown in Fig. 2, the centrifugal force of any incremental volume ΔV acts radially but has nothing to support it in this direction. Therefore it must be supported by tangential forces in the material adjacent to the radial faces A and B. In order for the elementary volume to be in equilibrium, the radial components of the tangential forces which act at right angles to the radial faces A and B must balance the centrifugal force F, as shown in Fig. 3. The tangential stress in the material produces an elongation in the circumference and a reduction in the radial thickness of the cylindrical shell. Hence in this case the centrifugal force is counterbalanced by tangential stresses which tend to increase the diameter of the annular cylinder. The percentage increase is greater for the internal diameter than for the external diameter on account
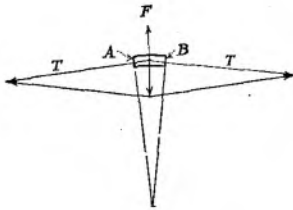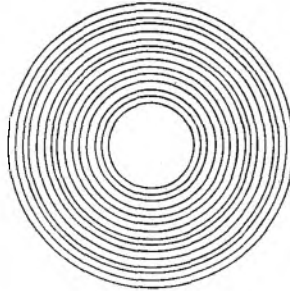

FIG. 3   FIG. 4

of the decrease in thickness of the cylinder walls. It is obvious, then, that if the rotor were composed of annular laminæ, as shown in Fig. 4, each ring would act independently in counterbalancing the centrifugal force acting on it. Since the centrifugal force and the mass of the laminæ vary directly as the radius, the tangential stresses in the laminæ increase approximately as the square of the radius. Similarly, the radial expansion is a maximum for the outside laminæ and decreases approximately as the square of the radius for the internal laminæ.

5 The solid or hollow cylindrical rotor of appreciable radial thickness is in reality a combination of the two cases previously considered. The centrifugal forces of the circumferential sections are the same as for the laminated rotor, but the sections are not free to expand independently and thereby counterbalance their own respective centrifugal forces. Since the centrifugal force of the cir-

cumferential section varies as the square of the radius and all of the sections are elastically coupled together, part of the centrifugal force of the outermost section will be transmitted to the adjacent inner section as an external radial load. This additional external load on the inner section produces radial and tangential stresses in it which add to the corresponding stresses due to its own centrifugal force. This section, in turn, transmits a greater external radial load to the next adjacent inner section than it would if the outermost section were not present. This loading-back effect cumulatively increases in passing from the external to the internal sections, and consequently the radial and tangential stresses are higher for the internal sections than for the external sections. It is obvious in the case of a solid rotor that the radial and tangential stresses increase from zero and minimum values, respectively, at the external radius to maximum and equal values at the center. Similarly, in the case of the hollow cylindrical rotor the tangential stress increases from a minimum value at the external radius to a maximum value at the internal radius. The radial stress increases from zero at the external radius and reaches a maximum value, and then decreases and reaches zero at the internal radius. In both cases the greatest stress occurs at the internal radius and consequently this is the point at which rupture or failure usually occurs. It is also obvious from the above analysis that with a central hole in the rotor the internal radial stress is zero and its equivalent effect must be compensated for by an increased tangential stress at this point. Since the two stresses are normally equal at the center and the radius and tangent of a center point are identical, a pinhole at the center would reduce the radial stress to zero and consequently double the tangential stress.

6   It is obvious from the above consideration, neglecting the stresses due to shear and bending, that any incremental volume, ΔV, of the rotor, Fig. 5, is subjected to the following forces due to rotation:

a   The centrifugal force of the elementary volume, which acts outward in the radial direction

b   The tangential forces which act at right angles to the radial faces A and B, and have radial components directed inward

c   Radial forces which tend to separate the elementary volume from the main body of the rotor at the inner and

outer circumferential surfaces of the elementary volume. The difference between these two forces determines the magnitude of the force acting on the elementary volume; and the relative magnitude of the forces determines the direction of the resultant. If the radial force at the outer surface is the greater, the direction of the resultant will be outward. Similarly, if the radial force at the inner surface is the greater, the resultant force acting on the elementary section will be directed inward. In order for the incremental volume to be in stable equilibrium, the algebraic sum of the radial components must be zero.
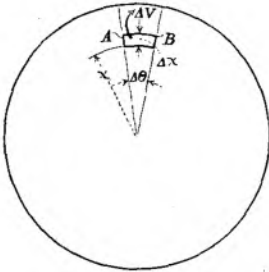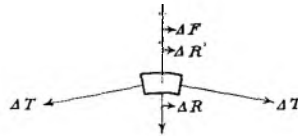

FIG. 5   FIG. 6

DERIVATION OF THE GENERAL EQUATIONS

7   In Figs. 5 and 6 let —

x = radial distance of the elementary section from the rotor axis

l = length of rotor (constant)

S_t = tangential stress acting at right angles to the radial faces of the elementary section

S_r = radial stress at the inner circumference of the section

W = weight per unit volume of the rotor material

g = acceleration due to gravity in feet per second per second

$\dot{w}$ = angular velocity of the rotor in radians per second

v = contraction coefficient (Poisson's ratio).

Then from Figs. 5 and 6,

ΔV $= x\Delta\theta l \cdot \Delta x$ = volume of the elementary section

ΔM $= \dfrac{W}{g} lx \cdot \Delta\theta \cdot \Delta x$ = mass of the elementary section

$\Delta F = \Delta M \times \dfrac{w^2 x}{12} =$ centrifugal force acting on the elementary section

$= \dfrac{W}{g} \, lw^2 \, \dfrac{x^2}{12} \, \Delta\theta \cdot \Delta x$, where x is in inches

$\Delta T = S_t l \Delta x =$ tangential force acting across the radial faces or A and B

$\Delta R = S_r lx \cdot \Delta\theta =$ radial force at the inner circumference of the section

$\Delta R' = (S_r + \Delta S_r) \cdot (x + \Delta x) \cdot l \cdot \Delta\theta =$ radial force at the outer circumference of the section

$\Delta Z =$ radial component of the tangential forces

$= 2\Delta T \sin \dfrac{\Delta\theta}{2}$

$= \Delta T \cdot \Delta\theta$ (approximately)

$= S_t \, l \cdot \Delta x \cdot \Delta\theta$

In order for the elementary section to be in equilibrium —

$$\Delta F + \Delta R' - \Delta R - \Delta Z = 0$$

or $\dfrac{W w^2 l x^2}{12g} \cdot \Delta\theta \cdot \Delta x + (S_r + \Delta S_r)\,(x + \Delta x)\, l \cdot \Delta\theta - S_r l\, x \cdot \Delta\theta$

$$- S_t l \cdot \Delta\theta \cdot \Delta x = 0$$

Simplifying this equation and neglecting the differentials of the second order, gives —

$$\frac{ds_r}{dx} + \frac{l}{x}(S_r - S_t) + \frac{W}{12g}w^2 x = 0 \quad \ldots \ldots \quad [1]$$

In order to solve this differential equation, it is convenient to express S_r and S_t in terms of the radial elongation.

Let $\lambda =$ total radial elongation of a point originally at a distance, x, from the axis, Figs. 7(b) and 7(c)

$\lambda + \Delta\lambda =$ total radial elongation of a point originally at a distance, $x + \Delta x$, from the axis, Figs. 7(b) and 7(c)

$E_r' =$ radial expansion per unit length of the radius

$E_t' =$ tangential expansion per unit length of the circumference.

The radial expansion due to S_r is $\dfrac{S_r}{E}$, where E is the modulus of elasticity of the rotor material. The tangential stress acting simul-

taneously produces a contraction $\dfrac{vS_t}{E}$. The resultant radial and tangential expansions per unit length due to the stresses S_r and S_t are —

$$\left.\begin{aligned} E_r' &= \frac{1}{E}\,(S_r - vS_t) \\[2mm] E_t' &= \frac{1}{E}\,(S_t - vS_r) \end{aligned}\right\} \quad \cdots \cdots \cdots \text{[2]}$$

The values of E_r and E_t can also be found in terms of the rotor dimensions by considering the length of the radius and circumference before and after expansion, as shown in Figs. 7(a), 7(b), and 7(c).

$$E_t' = \frac{2\pi(x + \lambda) - 2\pi x}{2\pi x} = \frac{\lambda}{x} \quad \cdots \cdots \text{[3]}$$
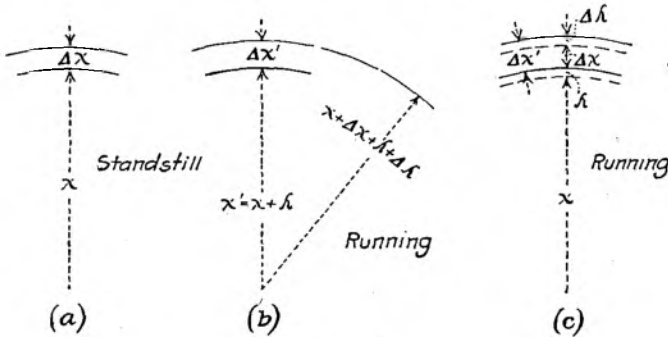

FIG. 7

8 From Fig. 7(c) the radial elongation of the length Δx is $\Delta\lambda$, hence the elongation per unit length —

$$E_r' = \frac{\Delta\lambda}{\Delta x} = \frac{d\lambda}{dx} \quad \cdots \cdots \cdots \text{[4]}$$

Substituting these values of E_r' and E_t' in Equations [2] and solving for S_r and S_t gives —

$$\left.\begin{aligned} S_r &= \frac{E}{1 - v^2}\left(\frac{v\lambda}{x} + \frac{d\lambda}{dx}\right) \\[2mm] S_t &= \frac{E}{1 - v^2}\left(\frac{\lambda}{x} + v\,\frac{d\lambda}{dx}\right) \end{aligned}\right\} \quad \cdots \cdots \cdots \text{[5]}$$

Taking the derivative of S_r with respect to x gives —

$$\frac{ds_r}{dx} = \frac{E}{1 - v^2}\left(\frac{vx\dfrac{d\lambda}{dx} - \lambda v}{x^2} + \frac{d^2\lambda}{dx^2}\right) \quad \cdots \cdots \text{[6]}$$

Substituting $\dfrac{ds_r}{dx}$, S_r, and S_t in Equation [1] and simplifying gives —

$$\frac{d^2\lambda}{dx^2} + \frac{1}{x}\frac{d\lambda}{dx} - \frac{\lambda}{x^2} + Ax = 0 \quad \ldots \ldots \quad [7]$$

where $A = \dfrac{W}{12g}\, w^2\, \dfrac{(1 - v^2)}{E}$.

The term involving x can be eliminated by substituting —

$$\lambda = Z + ax^3$$

$$\frac{d\lambda}{dx} = \frac{dZ}{dx} + 3\,ax^2, \text{and}\,\frac{d^2\lambda}{dx^2} = \frac{d^2Z}{dx^2} + 6\,ax$$

Substituting $\dfrac{d^2\lambda}{dx^2}$, $\dfrac{d\lambda}{dx}$ and λ in Equation [7] gives —

$$\frac{d^2Z}{dx^2} + \frac{1}{x}\frac{dZ}{dx} - \frac{Z}{x^2} + (8a + A)x = 0$$

or

$$\frac{d^2Z}{dx^2} + \frac{1}{x}\frac{dZ}{dx} - \frac{Z}{x^2} = 0 \quad \ldots \ldots \ldots \quad [8]$$

where

$$a = -\frac{A}{8} = \frac{-W}{12g}\, w^2\, \frac{(1 - v^2)}{8\,E}$$

To solve this differential equation, let —

$$Z = bx^n, \frac{dZ}{dx} = n\,bx^{n-1}, \text{and}\,\frac{d^2Z}{dx^2} = n(n-1)\,bx^{n-2}$$

Substituting $\dfrac{d^2Z}{dx^2}$ in Equation [8] and simplifying, gives —

$$n^2 = 1$$

therefore —

$$n = \pm\,1$$

and the general solution is —

$$Z = b_1 x + b_2 x^{-1}$$

Hence the general expression for the radial elongation is —

$$\lambda = ax^3 + b_1 x + b_2 x^{-1} \quad \ldots \ldots \ldots \quad [9]$$

$$\frac{\lambda}{x} = ax^2 + b_1 + b_2 x^{-2}$$

$$\frac{d\lambda}{dx} = 3ax^2 + b_1 - b_2 x^{-2}$$

Substituting $\dfrac{\lambda}{x}$ and $\dfrac{d\lambda}{dx}$ in Equations [5] and simplifying,

$$S_r = \frac{E}{1-v^2}\left(ax^2(3+v) + b_1(v+1) + b_2(v-1)x^{-2}\right) \Bigg\}$$
$$S_t = \frac{E}{1-v^2}\left(ax^2(3v+1) + b_1(v+1) - b_2(v-1)x^{-2}\right) \Bigg\}\ \ .\ \ [10]$$

9 The constants of integration b_1 and b_2 can be determined from the initial conditions for rotors of different specifications.

APPLICATION TO DIFFERENT TYPES OF CYLINDRICAL ROTORS

10 *Case I. Solid Cylindrical Rotor Without Central Hole —* The general equations for the radial and tangential stresses and the radial elongation in a cylindrical rotor due to the action of the
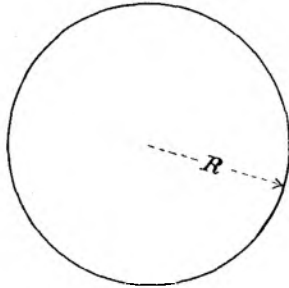

FIG. 8

centrifugal force are, from Equations [9] and [10],

$$S_r = \frac{E}{1-v^2}\left[ax^2(3+v) + b_1(1+v) + b_2(v-1)x^{-2}\right]$$
$$S_t = \frac{E}{1-v^2}\left[ax^2(3v+1) + b_1(1+v) - b_2(v-1)x^{-2}\right]$$
$$\lambda = ax^3 + b_1x + b_2x^{-1},$$

where S_r = radial stress in lb. per sq. in. at radius x

$\quad S_t$ = tangential stress in lb. per sq. in. at radius x

$\quad \lambda$ = radial elongation in inches at radius x

$\quad E = 29 \times 10^6$ = Young's modulus of elasticity

$\quad v = 0.3$ = Poisson's ratio, or contraction coefficient for steel

$\quad b_1, b_2$ = constants of integration

$$a = -\frac{W}{12g}\,w^2\,\frac{(1-v^2)}{8E}$$

$\quad W = 0.285$ lb. per cu. in. for the rotor material

$\quad g = 32.2$ = acceleration due to gravity in ft. per sec. per sec.

w = angular velocity in radians per sec.

x = radius in inches.

11   The constants of integration b_1 and b_2 can be determined from the limiting conditions: namely,

$S_r = 0$, when $x = R$, where R is the external radius, and

$\lambda = 0$, when $x = 0$.

Therefore b_2 must be zero if these conditions be true.   Applying these conditions to the above equations gives —

$$b_1(1 + v) = - aR^2(3 + v) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{[10b]}$$

$$\lambda = ax^3 + b_4 x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{[10c]}$$

From Equation [10b] —

$$b_1 = - aR^2 \frac{3 + v}{1 + v}$$

therefore —

$$\lambda = a \left[x^3 - R^2 x \frac{3 + v}{1 + v} \right]$$

But

$$a = \frac{- W}{12g} w^2 \frac{1 - v^2}{8E}$$

or

$$\lambda = \frac{W}{12g} w^2 \frac{1 - v^2}{8E} \left[R^2 x \frac{(3 + v)}{1 + v} - x^3 \right]$$

but $w = \dfrac{V_x}{x}$ where V_x = peripheral speed at radius x in inches per sec.

Then —

$$\lambda = \frac{W}{12g} \frac{V_x^2}{x^2} \frac{1 - v^2}{8E} \left[R^2 x \frac{3 + v}{1 + v} - x^3 \right]$$

$$= \frac{W}{12g} V_x^2 \frac{1 - v^2}{8E} \left[\frac{R^2}{x} \frac{3 + v}{1 + v} - x \right]$$

For steel —

$$\frac{W}{12g} = \frac{490}{1728} \times \frac{1}{12} \times \frac{1}{32.2}$$

$$E = 29 \times 10^6$$

$$v = 0.3$$

$V_x = 12 V_{fx}$, where V_{fx} is peripheral speed in ft. per sec. at radius x.

For $x = R$,

$$\lambda = \frac{490}{1728} \times \frac{1}{12} \times \frac{1}{32.2} \frac{(1 - 0.3^2)}{8 \times 29 \times 10^6} \times 12^2 \left[\frac{3 + 0.3}{1 + 0.3} - 1 \right] R V_{fR}^2$$

$$\lambda = 0.64 \times 10^{-9} \, RV_{fr}^2,$$

where R = external radius in inches and V_{fr} = peripheral speed in ft. per sec. at R. For any value of x, the radial elongation is —

$$\lambda = 0.64 \times 10^{-9} \, V_{fr}^2 \, x \left(1.65 - 0.65 \, \frac{x^2}{R^2}\right)$$

12 Substituting these values of b_1, b_2 and a in Equations [1] and [2],

$$S_t = 0.043606 \, V_{fr}^2 \left(1 - 0.57575 \, \frac{x^2}{R^2}\right) \text{ at any radius } x$$
$$S_t = 0.0185 \, V_{fr}^2 \text{ at the external radius, i.e., } x = R \qquad \cdots [11]$$
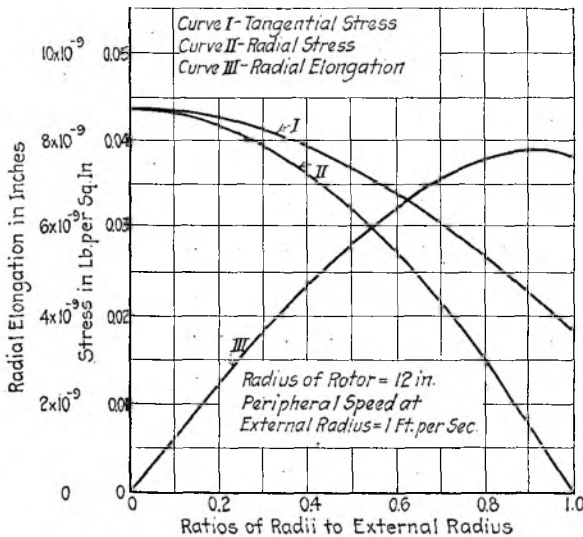$$S_t = 0.043606 \, V_{fr}^2 \text{ at center of disk, i.e., } x = 0$$

FIG. 9 STRESSES AND RADIAL ELONGATION OF SOLID CYLINDRICAL ROTOR

$$S_r = 0.043606 \, V_{fr}^2 \left(1 - \frac{x^2}{R^2}\right) \text{ at any radius } x$$
$$S_r = 0 \text{ at the external radius, i.e. } x = R \qquad \cdots [12]$$
$$S_r = 0.043606 \, V_{fr}^2 \text{ at center of disk, i.e., } x = 0$$

where V_{fr} is the peripheral speed in ft. per sec. at the external radius, and both x and R are expressed in inches.

13 Curves I and II of Fig. 9 show the values of the tangential and radial stresses in a solid cylindrical rotor for a peripheral speed,

at the external radius, of 1 ft. per sec. The stress is expressed in lb. per sq. in. and the radii are expressed as decimal fractions of the external radius. Both stresses are a maximum and equal at the axis of the rotor. Curve III shows the radial elongation in inches for a solid cylindrical rotor which has a radius of 12 in. and an external peripheral speed of 1 ft. per sec. The elongation reaches a maximum value at a radius equal to approximately 92 per cent of the external radius. The decrease in the radial elongation for radii greater than 92 per cent of the external radius is due to the fact that the radial contraction of the material at these radii caused by the tangential stress is greater than the radial expansion produced by the radial stress.

14   NUMERICAL EXAMPLE:

$$R = 20 \text{ in. external radius}$$
$$V_{fr} = 400 \text{ ft. per sec. at the external radius}$$
$$\lambda = 0.64 \times 10^{-9} \times 20 \times 400^2$$
$$= 0.00205 \text{ in. elongation at the external radius.}$$

The stresses at the center of the rotor are

$$S_t = S_r = 0.043606 \times 400^2 = 6975 \text{ lb. per sq. in.}$$

The stresses at the external radius are —

$$S_r = 0$$
$$S_t = 0.0185 \times 400_2 = 2960 \text{ lb. per sq. in.}$$

15   *Case II. Cylindrical Rotor with Central Hole* (Fig. 10). — This case applies particularly to rotating cylindrical drums, plate flywheels, and pulley rims. The limiting conditions are —

$$S_r = 0 \text{ when } x = r, \text{ and}$$
$$S_r = 0 \text{ when } x = R$$

Applying these conditions to Equation [10] gives —

$$0 = ar^2 (3 + v) + b_1 (v + 1) + \frac{b_2}{n^2} (v - 1)$$

$$0 = aR^2 (3 + v) + b_1 (v + 1) + \frac{b_2}{R^2}(v - 1)$$

Solving these simultaneous equations for b_1 and b_2 gives —

$$b_1 = \frac{- a(3 + v) (r^2 + R^2)}{v + 1}$$

$$b_2 = \frac{a(3 + v) r^2 R^2}{v - 1}$$

16   The values of the stresses and radial elongation can be found by substituting the values of b_1 and b_2 in Equations [9] and [10].

$$S_t = 0.043606\ V_{fr}^2 \left[\frac{r^2}{R^2} + 1 + \frac{r^2}{x^2} - \frac{x^2}{R^2}0.57575\right] \text{at any radius } x\ [\mathbf{13}]$$

$$S_t = 0.043606\ V_{fr}^2 \left[0.42424\frac{r^2}{R^2} + 2\right] \text{at } x = r \quad \ldots \ldots [\mathbf{14}]$$

$$S_t = 0.043606\ V_{fr}^2 \left[\frac{2r^2}{R^2} + 0.42424\right] \text{at } x = R \ldots \ldots \ldots [\mathbf{15}]$$

$$S_r = 0.043606\ V_{fr}^2 \left[\frac{r^2}{R^2} + 1 - \frac{x^2}{R^2} - \frac{r^2}{x^2}\right] \text{at any radius } x \quad . \ .[\mathbf{16}]$$

$$S_r = 0 \text{ at } x = r \text{ and } x = R \ldots \ldots \ldots \ldots \ldots \ldots [\mathbf{17}]$$

$$\lambda_R = 0.64 \times 10^{-9}\ V_{fr}^2\ R \left[1 + 4.71\frac{r^2}{R^2}\right] \text{at } x = R \ . \ . \ . \ .[\mathbf{18}]$$
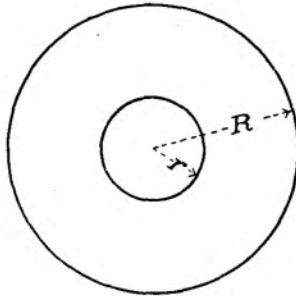

Fig. 10

$$\lambda_x = 0.64 \times 10^{-9}\ V_{fr}^2\ x \left[1.65 + 1.65\frac{r^2}{R^2} - 0.65\frac{x^2}{R^2} + 3.06\frac{r^2}{x^2}\right]$$
$$\text{at } x = x \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . [\mathbf{19}]$$

$$\lambda_r = 0.64 \times 10^{-9}\ V_{fr}^2\ r \left[4.71 + \frac{r^2}{R^2}\right] \text{at } x = r \ . \ . \ . \ . \ .[\mathbf{19a}]$$

17   The curves in Fig. 12 show the variation of the radial and tangential stresses for hollow cylindrical rotors with different ratios of internal to external diameters. The stresses are based on a peripheral speed at the external surface of 1 ft. per sec. The radial elongation is based on a rotor with a 12-in. radius and a peripheral speed at the external radius of 1 ft. per sec.

18   NUMERICAL EXAMPLE:

   R = 20 in., external diameter of rotor
   r = 12 in., internal diameter of rotor
   $\frac{r}{R}$ = 0.6, ratio of internal to external diameter
   V_{fr} = 400 ft. per sec., peripheral speed at external radius.

From Figs. 11 and 12 the stresses for a rotor with an external peripheral speed of 1 ft. per sec. are —

0.05 lb. per sq. in. tangential stress at the external radius
0.094 lb. per. sq. in. tangential stress at the internal radius and
0.007 lb. per. sq. in. maximum radial stress at 0.8 radius.

Since the stresses vary as the square of the external peripheral speed the stresses for an external peripheral speed of 400 ft. per sec. are —

0.05 $\times$ 400^2 = 8000 lb. per sq. in. tangential stress at the external radius
0.094 $\times$ 400^2 = 15040 lb. per sq. in. tangential stress at the internal radius
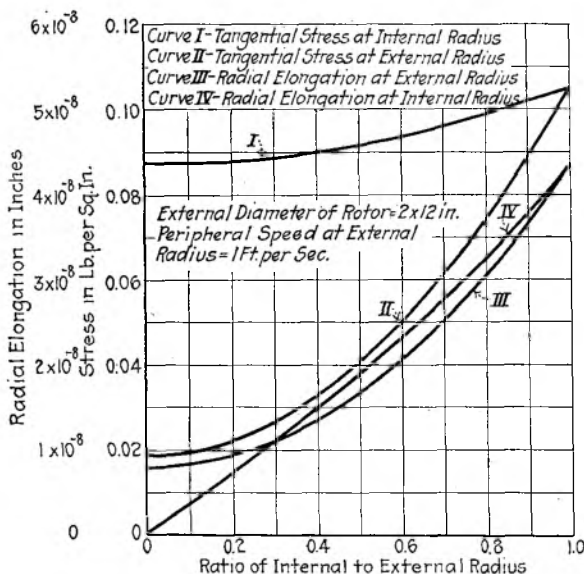0.007 $\times$ 400^2 = 1120 lb. per sq. in. maximum radial stress at 0.8 radius.

FIG. 11 STRESSES AND RADIAL ELONGATION OF HOLLOW CYLINDRICAL ROTOR

From Fig. 11 the total elongation in the radius for a rotor with a 12-in. radius and an external peripheral speed of 1 ft. per sec. is 2.075×10^{-8} in. Since the radial elongation is proportional to square of the external peripheral speed and the first power of the radius, the elongation in the radius of the rotor under consideration is $2.075 \times 10^{-8} \times 400^2 \times \frac{20}{12}$ or 0.005536 in., and the increase in the diameter is 2×0.005536 or 0.011072 in.

19 *Case III. Cylindrical Rotor with Central Hole and Externally Loaded* (Fig. 13). — This case applies to practically all rotors of both solid and laminated construction that have uniformly distributed windings imbedded in slots at the periphery.

20   Let —

S_{ro} = specific radial stress at external circumference due to external load

S_t = tangential stress across radial plane at distance x from axis

S_r = radial stress at distance x from neutral axis.

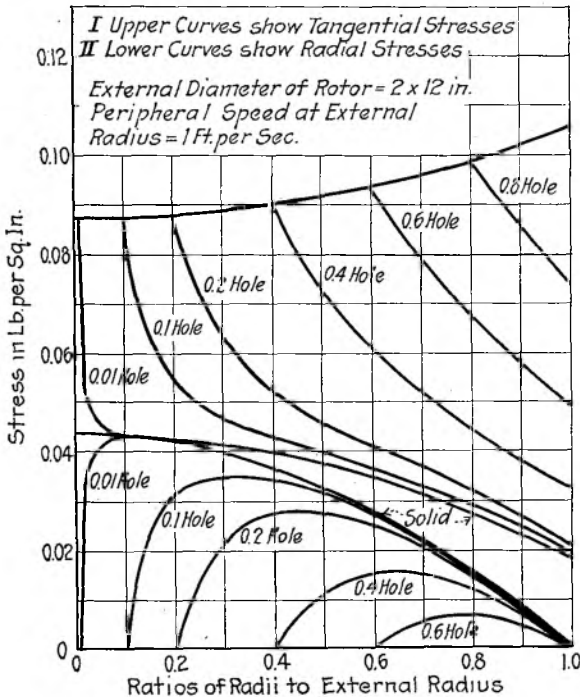S_t and S_r include the loading-back effect of the external load which

FIG. 12 RADIAL AND TANGENTIAL STRESSES OF HOLLOW CYLINDRICAL ROTOR

is transmitted to the elementary section. Hence, the general formulas derived for the rotating cylinder apply equally well in this case, when the proper constants of integration are determined.

21   The conditions in this case are —

$$S_r = S_{ro} \text{ when } x = R \atop S_r = 0 \quad \text{ when } x = r \quad \right\} \quad \cdots \cdots \cdots \quad [20]$$

Hence b_1 and b_2 can be determined by substituting these limiting conditions in Equation [10], which gives —

$$\left.\begin{array}{l} \dfrac{1 - v^2}{E} S_{ro} = aR^2 (3 + v) + b_1 (v + 1) - \dfrac{b_2}{R^2} (1 - v) \\[2mm] 0 = ar^2 (3 + v) + b_1 (v + 1) - \dfrac{b_2}{r^2} (1 - v) \end{array}\right\} \quad . \quad [21]$$

From which —

$$\left.\begin{array}{l} b_1 = -\dfrac{a(3 + v)(r^2 + R^2)}{1 + v} + \dfrac{S_{ro}(1 - v)R^2}{E(R^2 - r^2)} \\[3mm] b_2 = -\dfrac{a(3 + v)r^2R^2}{1 - v} + \dfrac{S_{ro}(1 + v)r^2R^2}{E(R^2 - r^2)} \end{array}\right\} \quad \cdots \quad [22]$$
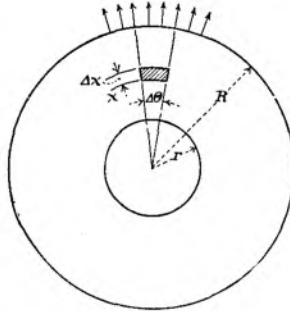

FIG. 13

Substituting b_1 and b_2 in Equations [10] and simplifying, gives —

$$S_r = 0.043606 \, V_{fr}^2 \left[\frac{r^2}{R^2} + 1 - \frac{x^2}{R^2} - \frac{r^2}{x^2}\right] + S_{ro}\left[\frac{1 - \dfrac{r^2}{x^2}}{1 - \dfrac{r^2}{R^2}}\right] \quad \cdots \quad [23]$$

$$S_t = 0.043606 \, V_{fr}^2 \left[\frac{r^2}{R^2} + 1 + \frac{r^2}{x^2} - 0.57575\frac{x^2}{R^2}\right] + S_{ro}\left[\frac{1 + \dfrac{r^2}{x^2}}{1 - \dfrac{r^2}{R^2}}\right] \quad [24]$$

When $x = R$,

$$S_t = 0.043606 \, V_{fr}^2 \left[2\frac{r^2}{R^2} + 0.42424\right] + S_{ro}\left[\frac{1 + \dfrac{r^2}{R^2}}{1 - \dfrac{r^2}{R^2}}\right] \quad \cdots \quad [25]$$

When $x = r$,

$$S_t = 0.043606 \, V_{fr}^2 \left[2 + 0.042424\frac{r^2}{R^2}\right] + 2S_{ro}\left[\frac{1}{1 - \dfrac{r^2}{R^2}}\right] \quad \cdots \quad [26]$$

To find the radial elongation —

$$\lambda = ax^3 + b_1 x + b_2 x^{-1}$$

Substituting the values of b_1 and b_2 from Equation [22] and evaluating a gives —

$$\lambda_x = ax^3 - \frac{a(3+v)\,(r^2+R^2)x}{1+v} - \frac{a(3+v)r^2R^2}{x(1-v)} + \frac{S_{ro}(1-v)R^2x}{E(R^2-r^2)}$$

$$+ \frac{S_{ro}(1+v)r^2R^2}{E(R^2-r^2)x}$$

$$= 0.64 \times 10^{-9}\, V_{fr}^2 x \left[1.65 + 1.65\,\frac{x^2}{R^2} - 0.65\,\frac{x^2}{R^2} \right.$$

$$\left. + 3.06\,\frac{r^2}{x^2} \right] + 34.5 \times 10^{-9}\, S_{ro}x \left[\frac{0.7+1.3\dfrac{r^2}{x^2}}{1-\dfrac{r^2}{R^2}} \right] \quad . \quad . \quad . \quad [27]$$

When $x = R$,

$$\lambda_R = 0.64 \times 10^{-9}\, V_{fr}^2 R \left[1 + 4.71\,\frac{r^2}{R^2} \right]$$

$$+ 34.5 \times 10^{-9}\, S_{ro}R \left[\frac{0.7+1.3\dfrac{r^2}{R^2}}{1-\dfrac{r^2}{R^2}} \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [28]$$

$$\lambda_r = 0.64 \times 10^{-9}\, V_{fr}^2\, r \left[4.71 + \frac{r^2}{R^2} \right] + 34.5 \times 10^{-9}\, S_{ro}r \left[\frac{2}{1-\dfrac{r^2}{R^2}} \right] \, [28a]$$

22 It is evident from the above equations that the effect of the external radial load is to add a variable term to the original equations for the stresses and radial elongation of hollow cylindrical rotor without an external load. The curves in Fig. 14 show the additional tangential stresses at the external and internal radii, and the total radial elongation of cylindrical rotors with different ratios of internal to external radii. The curves in Fig. 15 show the variation of the radial stress due to the external load for cylindrical rotors having different ratios of internal to external radii. The stresses are based on an external radial load of 1 lb. per sq. in. and the radial elongations are based on a rotor with a 12-in. external radius and an external load of 1 lb. per sq. in. Hence, in considering the same numerical example previously considered for the hollow cylindrical rotor, except that it has an uniform external radial load of 2000 lb. per sq. in. the additional terms are:

From the curves in Figs. 14 and 15,

    2.13 lb. per sq. in. = tangential stress at the external radius
    3.13 lb. per sq. in. = tangential stress at the internal radius
    0.683 lb. per sq. in. = radial stress at 0.8 radius.

Since the stresses vary directly as the external radial load,

    2000×2.13 = 4260 lb. per sq. in. = tangential stress at the external radius
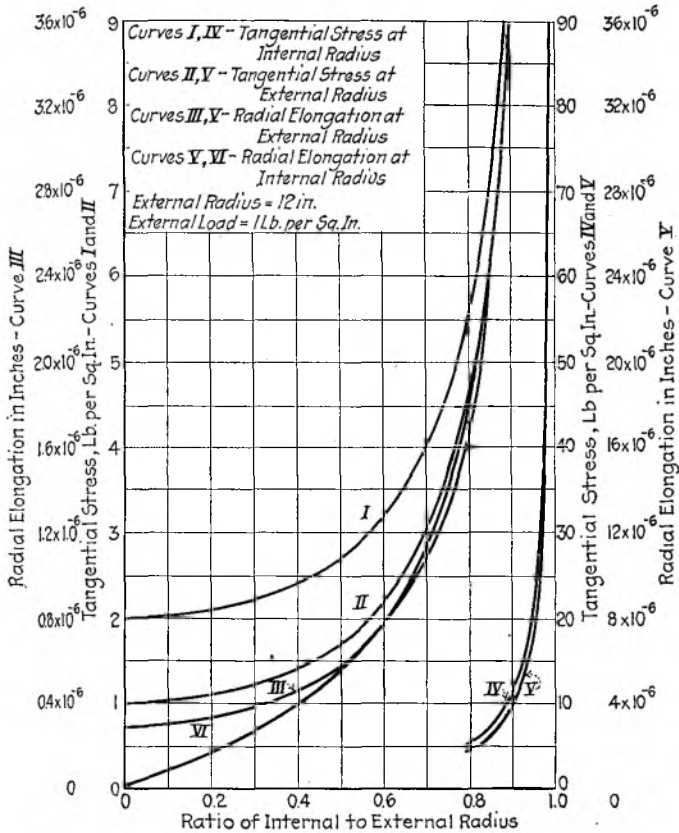

FIG. 14 ADDITIONAL TANGENTIAL STRESSES AND RADIAL ELONGATION OF A
HOLLOW CYLINDRICAL ROTOR DUE TO EXTERNAL LOAD

    2000×3.13 = 6260 lb. per sq. in. = tangential stress at the internal radius
    2000×0.683 = 1366 lb. per sq. in. = radial stress at 0.8 radius.

Then the total stresses are —

    $8000 + 4260$ = 12260 lb. per sq. in. = tangential stress at external radius
    $15040 + 6260$ = 21300 lb. per sq. in. = tangential stress at internal radius
    $1120 + 1366$ = 2486 lb. per sq. in. = radial stress at 0.8 radius.

23   From curve III of Fig. 14, the radial elongation of a rotor with a 12-in. external radius and an external radial load of 1 lb. per sq. in. is 0.76×10^{-6} in. Then the radial elongation for a rotor with a 20-in. radius and an external load of 2000 lb. per sq. in. is $0.76 \times 10^{-6} \times 2000 \times \dfrac{20}{12}$ or 0.002540 in. The total radial elongation at the external radius is $0.00254 + .005536$ or 0.00808 in.

24   *Case IV.   Solid Cylindrical Rotor without Central Hole but*
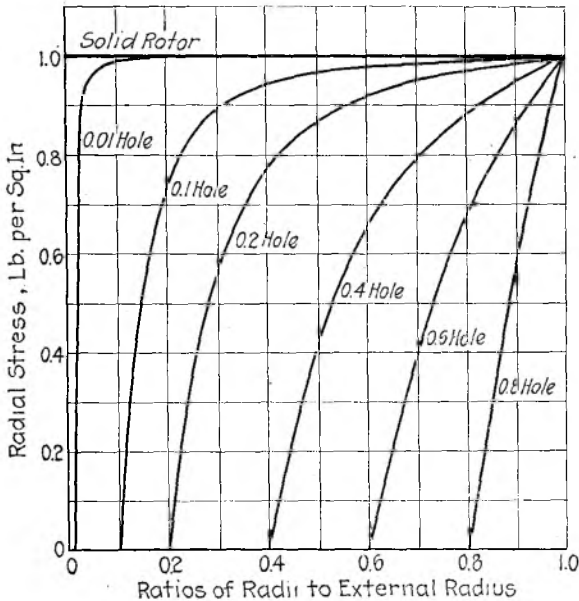
FIG. 15 VARIATION OF RADIAL STRESS DUE TO AN EXTERNAL LOAD FOR HOLLOW CYLINDRICAL ROTORS OF DIFFERENT RATIOS OF INTERNAL TO EXTERNAL DIAMETER

(External radial load = 1 lb. per sq. in.)

Externally Loaded   (Fig. 16). — Refer to Fig. 13.   In this case the limiting conditions are:

$$S_r = S_{ro} \text{ when } x = R$$

$$\lambda = 0 \text{ when } x = 0$$

since —

$$\lambda = ax^3 + b_1 x + b_2 x^{-1}$$

when —

$$x = 0, \lambda = 0. \quad \text{Hence } b_2 = 0$$

Substituting the condition that $S_r = S_{ro}$ when $x = R$ in Equation [**10**] gives —

$$\frac{1 - v^2}{E} S_{ro} = aR^2(3 + v) + b_1(v + 1)$$

and

$$b_1(v + 1) = \frac{1 - v^2}{E} S_{ro} - aR^2(3 + v)$$

$$\cdots \quad [\mathbf{29}]$$

Substituting $b_1(v + 1)$ in Equation [**10**] and simplifying, gives —

$$S_r = \frac{E}{1 - v^2}\left[ax^2(3 + v) - aR^2(3 + v) \right] + S_{ro}$$

$$= \frac{Ea(3 + v)}{1 - v^2}\left[x^2 - R^2 \right] + S_{ro}$$
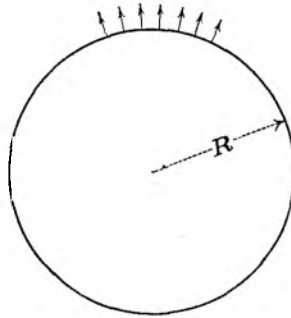
FIG. 16

$$= -\, 0.043606\, V_{fr}^2\left[\frac{R^2}{x^2} - 1\right] + S_{ro}$$

$$= 0.043606\, V_{fr}^2\left[1 - \frac{x^2}{R^2}\right] + S_{ro} \text{ at any radius } x \quad \cdots \quad [\mathbf{30}]$$

and —

$$S_t = 0.043606\, V_{fr}^2\left[1 - 0.57575\,\frac{x^2}{R^2}\right] + S_{ro} \text{ at any radius } x \quad [\mathbf{31}]$$

when —

$$x = R$$
$$S_r = S_{ro}$$

and —

$$S_t = 0.043606\, V_{fr}^2 \times 0.42424 + S_{ro}$$

$$= 0.0185\, V_{fr}^2 + S_{ro} \quad \cdots \cdots \cdots \cdots \cdots \quad [\mathbf{32}]$$

And when $x = 0$,

$$S_r = S_t = 0.043606 \, V_{fr}^2 + S_{ro} \quad \ldots \ldots \ldots \ldots \quad [33]$$

Since $b_1(v + 1) = \dfrac{1 - v^2}{E} S_{ro} - aR^2(3 + v)$

$$b_1 = \frac{1 - v}{E} S_{ro} - aR^2 \frac{3 + v}{1 + v}$$

$$b_2 = 0$$

therefore —

$$\lambda = ax^3 + b_1 x$$

$$= ax^3 + \left[\frac{(1 - v)S_{ro}}{E} - \frac{aR^2(3 + v)}{1 + v} \right] x$$

If $x = R$,

$$\lambda = 0.64 \times 10^{-9} \, V_{fr}^2 R + 24.1 \times 10^{-9} \, S_{ro} R \quad \ldots \ldots \quad [34]$$
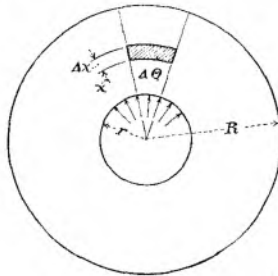

FIG. 17

25   Thus in the case of the solid rotor with an external radial load, the superimposed tangential and radial stresses at any fiber are constant and equal in magnitude to the radial stress at the outer radius due to the external load. If the solid rotor of the example previously considered has an external radial load of 2000 lb. per sq. in., the total stresses are —

6975 + 2000 = 8975 lb. per sq. in. tangential and radial stresses at the center of the rotor

2960 + 2000 = 4960 lb. per sq. in. tangential stress at the external radius.

26   The elongation in the radius due to the external load is 24.1×10^{-9} $\times 2000 \times 20 = 0.000964$ in., and the total radial elongation is $0.00205 + 0.000964 = 0.003014$ in.

27   *Case V. Cylindrical Rotor with Central Hole and Internally Loaded* (Fig. 17). — A conspicuous example of this class is the coil retaining ring which supports the end turns of the field winding of a turbo rotor. The limiting conditions in this case are —

$$\left. \begin{array}{l} S_r = 0 \text{ when } x = R \\ S_r = -S_{ro}' \text{ when } x = r \end{array} \right\} \quad \ldots \ldots \ldots \quad [35]$$

where S_{ro}' = radial compressive stress at the internal radius.   Substituting these conditions in Equation [10] gives —

$$\left.\begin{array}{l} -\dfrac{1-v^2}{E}S_{ro}' = ar^2(3+v) + b_1(1+v) - \dfrac{b_2}{r^2}(1-v) \\[3mm] \qquad 0 = aR^2(3+v) + b_1(v+1) - (1-v)\dfrac{b_2}{R^2} \end{array}\right\} \quad . \,[36]$$

28   Solving these simultaneous equations for b_1 and b_2 and substituting in Equation [10] gives —

$$S_t = 0.043606\,V_f r^2\left[\frac{r^2}{R^2} + \frac{r^2}{x^2} + 1 - 0.57575\,\frac{x^2}{R^2}\right]$$

$$+\,S_{ro}'\left[\frac{\dfrac{r^2}{R^2}+\dfrac{r^2}{x^2}}{1-\dfrac{r^2}{R^2}}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad [37]$$

$$S_r = 0.043606\,V_f r^2\left[\frac{r^2}{R^2} - \frac{r^2}{x^2} - \frac{x^2}{R^2} + 1\right] - S_{ro}'\left[\frac{\dfrac{r^2}{x^2}-\dfrac{r^2}{R^2}}{1-\dfrac{r^2}{R^2}}\right] \quad . \quad [38]$$

When $x = R$, $S_r = 0$, and —

$$S_t = 0.043606\,V_f r^2\left[0.42424 + 2\,\frac{r^2}{R^2}\right] + 2S_{ro}'\,\frac{\dfrac{r^2}{R^2}}{1-\dfrac{r^2}{R^2}} \quad . \quad . \quad . \quad [39]$$

When $x = r$, $S_r = -S_{ro}'$, and —

$$S_t = 0.043606\,V_f r^2\left[2 + 0.42424\,\frac{r^2}{R^2}\right] + S_{ro}'\left[\frac{1+\dfrac{r^2}{R^2}}{1-\dfrac{r^2}{R^2}}\right] \quad . \quad . \quad [40]$$

29   In order to get an expression for the radial elongation, Equation [36] can be solved for b_1 and b_2:

$$b_1 = \frac{S_{ro}'(1-v)\,r^2}{E\,(R^2-r^2)} - \frac{a(3+v)\,(r^2+R^2)}{1+v}$$

$$b_2 = \frac{S_{ro}'(1+v)\,R^2r^2}{E\,(R^2-r^2)} - \frac{a(3+v)\,R^2r^2}{1-v}$$

Hence —

$$\begin{aligned}
\lambda &= ax^3 + b_1x + b_2x^{-1} \\[2mm]
&= ax^3 + \frac{S_{ro}'(1-v)r^2}{E\,(R^2-r^2)}\,x - \frac{a(3+v)\,(r^2+R^2)}{1+v}\,x \\[2mm]
&\quad + \frac{S_{ro}'(1+v)\,R^2r^2}{E\,(R^2-r^2)\,x} - \frac{a(3+v)\,(R^2r^2)}{(1-v)\,x}
\end{aligned}$$

$$\lambda = 0.64 \times 10^{-9} V_{fr}^2 x \left[1.65 + 1.65 \frac{r^2}{R^2} - 0.65 \frac{x^2}{R^2} + 3.06 \frac{r^2}{x^2} \right]$$

$$+ 34.5 \times 10^{-9} S_{ro}' x \left[\frac{0.7 \frac{r^2}{R^2} + 1.3 \frac{r^2}{x^2}}{1 - \frac{r^2}{R^2}} \right] \quad \ldots \quad [41]$$

The total radial elongation at the external radius is—

$$\lambda = 0.64 \times 10^{-9} V_{fr}^2 R \left[1 + 4.71 \frac{r^2}{R^2} \right] + 69 \times 10^{-9} S_{ro}' R \left[\frac{\frac{r^2}{R^2}}{1 - \frac{r^2}{R^2}} \right] \quad [41a]$$

The total radial elongation at the internal radius is—

$$\lambda_r = 0.64 \times 10^{-9} \times V_{fr}^2 r \left[4.71 + \frac{r^2}{R^2} \right]$$

$$+ 34.5 \times 10^{-9} S_{ro}' r \left[\frac{1.3 + 0.7 \frac{r^2}{R^2}}{1 - \frac{r^2}{R^2}} \right] \quad \ldots \ldots \ldots \quad [41b]$$

30   The curves in Fig. 18 show the additional tangential stresses at the internal and external radii, and the total radial elongation for hollow cylindrical rotors with an internal radial load of 1 lb. per sq. in.   The variation of the radial stresses for different ratios of internal to external diameter are shown in Fig. 19.   If the rotor in the example given under Case III has an internal radial load of 2000 lb. per sq. in. instead of the external radial load, the additional stresses and radial elongation would be as follows:

31   From the curves in Figs. 18 and 19 the additional stresses for an internal load of 1 lb. per sq. in. are —

2.125 lb. per sq. in. tangential stress at the internal radius
1.125 lb. per sq. in. tangential stress at the external radius
− 0.315 lb. per sq. in. radial stress (compression) at 0.8 radius
0.475 × 10^{-6} in. radial elongation for rotor with a 12-in. radius.

Then for an internal radial load of 2000 lb. per sq. in.,

2000 × 2.125 = 4250 lb. per sq. in. tangential stress at the internal radius
2000 × 1.125 = 2250 lb. per sq. in. tangential stress at the external radius
− 2000 × 0.315 = − 630 lb. per sq. in. radial stress at 0.8 radius

$$2000 \times 0.475 \times 10^{-6} \times \frac{20}{12} = 0.00158 \text{ in. elongation in the radius.}$$

32   Using the results of the problem in Case II the total stresses and radial elongation are —

15040 + 4250 = 19290 lb. per sq. in. total tangential stress at the internal radius
  8000 + 2250 = 10250 lb. per sq. in. total tangential stress at the external radius
  1120 − 630 = 490 lb. per sq. in. radial stress at 0.8 radius
0.005536 + 0.00158 = 0.00712 in. total radial elongation at the external radius.

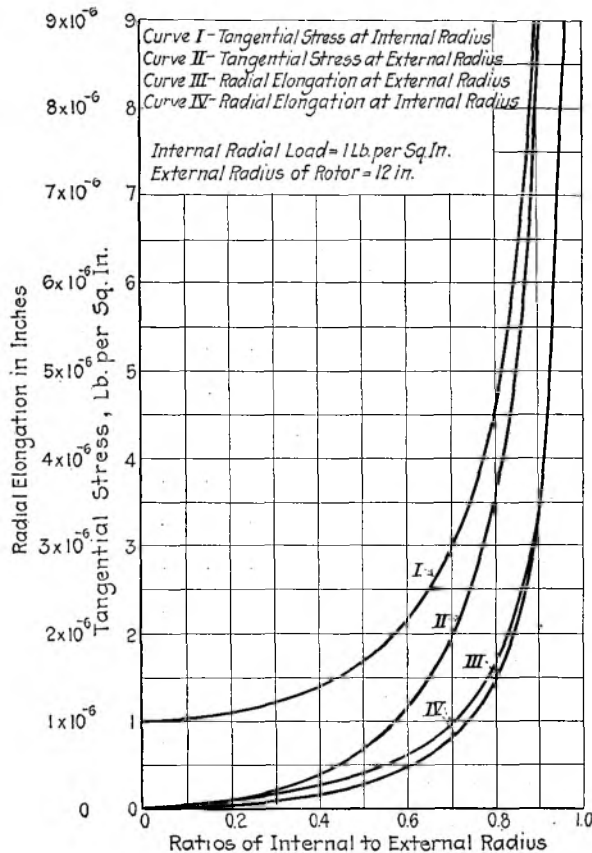FIG. 18 ADDITIONAL TANGENTIAL STRESSES AND RADIAL ELONGATION OF HOLLOW CYLINDRICAL ROTOR WITH UNIFORM INTERNAL RADIAL LOAD

33   *Case VI. Cylindrical Rotor with Central Hole and Loaded Internally and Externally* (Fig. 20). — Let S_{ro} be the radial stress at the outer circumference due to the external load, and $- S_{ro}'$ the radial stress at inner circumference due to internal load. The limiting conditions are —

$$S_r = S_{ro} \text{ when } x = R$$
$$S_r = -S_{ro}' \text{ when } x = r$$

34 Using these relations in Equation [10] gives

$$
\left.
\begin{aligned}
\frac{1 - v^2}{E} S_{ro} &= aR^2 (3 + v) + b_1 (v + 1) - \frac{b_2}{R^2}(1 - v) \\
-\frac{1 - v^2}{E} S_{ro}' &= ar^2 (3 + v) + b_1 (v + 1) - \frac{b_2}{r^2} (1 - v)
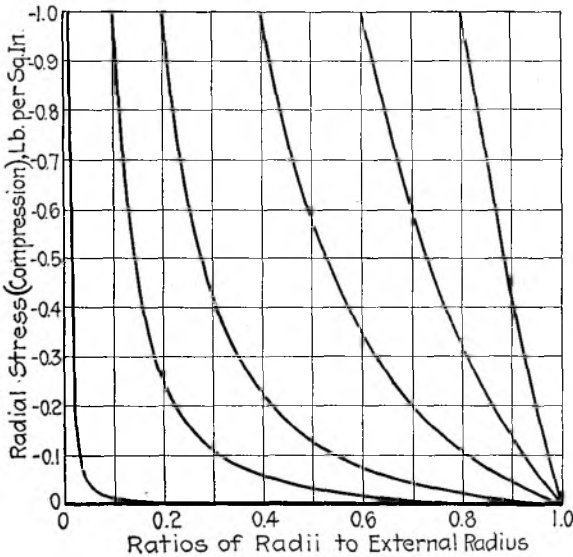\end{aligned}
\right\} \quad \dots \dots [42]
$$

FIG. 19 VARIATION OF RADIAL STRESS DUE TO AN INTERNAL LOAD FOR HOL-
LOW CYLINDRICAL ROTORS HAVING DIFFERENT RATIOS OF INTERNAL TO
EXTERNAL DIAMETER

(Internal radial load = 1 lb. per sq. in.)

Solving the two simultaneous Equations [42] for b_1 and b_2 gives —

$$
\left.
\begin{aligned}
b_1 &= \frac{(1 - v)\,(S_{ro}R^2 + S_{ro}'r^2)}{E\,(R^2 - r^2)} - \frac{a(3 + v)\,(R^2 + r^2)}{1 + v} \\
b_2 &= \frac{(1 + v)\,(S_{ro} + S'_{ro})\,R^2\,r^2}{E\,(R^2 - r^2)} - \frac{a(3 + v)\,R^2\,r^2}{1 - v}
\end{aligned}
\right\} \quad \dots [43]
$$

Substituting these values of b_1 and b_2 in Equations [10] gives the
radial and tangential stresses at any fiber.

$$S_r = 0.043606 V_f r^2 \left[\frac{r^2}{R^2} + 1 - \frac{x^2}{R^2} - \frac{r^2}{x^2} \right] + S_{ro} \left[\frac{1 - \dfrac{r^2}{x^2}}{1 - \dfrac{r^2}{R^2}} \right]$$

$$+ S_{ro}' \left[\frac{\dfrac{r^2}{R^2} - \dfrac{r^2}{x^2}}{1 - \dfrac{r^2}{R^2}} \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [44]$$

$$S_t = 0.043606 V_f r^2 \left[\frac{r^2}{R^2} + 1 + \frac{r^2}{x^2} - 0.57575 \frac{x^2}{R^2} \right] + S_{ro} \left[\frac{1 + \dfrac{r^2}{x^2}}{1 - \dfrac{r^2}{R^2}} \right]$$
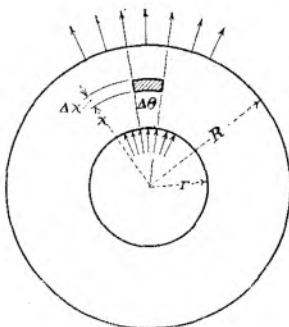

Fig. 20

$$+ S_{ro}' \left[\frac{\dfrac{r^2}{R^2} + \dfrac{r^2}{x^2}}{1 - \dfrac{r^2}{R^2}} \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [45]$$

35   The stresses at the external radius are —

$$S_r = S_{ro} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [46]$$

$$S_t = 0.043606 V_f r^2 \left[0.42424 + 2 \frac{r^2}{R^2} \right] + S_{ro} \left[\frac{1 + \dfrac{r^2}{R^2}}{1 - \dfrac{r^2}{R^2}} \right]$$

$$+ 2 S_{ro}' \left[\frac{\dfrac{r^2}{R^2}}{1 - \dfrac{r^2}{R^2}} \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [47]$$

36 Similarly, the values of the stresses at the internal radius are —

$$S_r = -S_{ro}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [48]$$

$$S_t = 0.043606 V_{fr}{}^2 \left[2 + 0.42424 \frac{r^2}{R^2} \right] + 2 S_{ro} \left[\frac{1}{1 - \frac{r^2}{R^2}} \right]$$

$$+ S_{ro}' \left[\frac{1 + \frac{r_2}{R^2}}{1 - \frac{r^2}{R^2}} \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [49]$$

37 The radial elongation at any fiber, from Equation [9], is —

$$\lambda_x = 0.64 \times 10^{-9} \, V_{fr}{}^2 x \left[1.65 + 1.65 \frac{r^2}{R^2} - 0.65 \frac{x^2}{R^2} + 3.06 \frac{r^2}{x^2} \right]$$

$$+ 34.5 \times 10^{-9} S_{ro} x \left[\frac{0.7 + 1.3 \frac{r^2}{x}}{1 - \frac{r^2}{R^2}} \right]$$

$$+ 34.5 \times 10^{-9} S_{ro}' x \left[\frac{0.7 \frac{r^2}{R^2} + 1.3 \frac{r^2}{x^2}}{1 - \frac{r^2}{R^2}} \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad [50]$$

When $x = R$,

$$\lambda_R = 0.64 \times 10^{-9} V_{fr}{}^2 R \left[1.0 + 4.71 \frac{r^2}{R^2} \right]$$

$$+ 34.5 \times 10^{-9} S_{ro} R \left[\frac{0.7 + 1.3 \frac{r^2}{R^2}}{1 - \frac{r^2}{R^2}} \right]$$

$$+ 69 \times 10^{-9} S_{ro}' R \left[\frac{\frac{r^2}{R^2}}{1 - \frac{r^2}{R^2}} \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [51]$$

When $x = r$,

$$\lambda_r = 0.64 \times 10^{-9} V_{fr}{}^2 r \left[4.71 + \frac{r^2}{R^2} \right] + 34.5 \times 10^{-9} S_{ro} r \left[\frac{2}{1 - \frac{r^2}{R^2}} \right]$$

$$+ 34.5 \times 10^{-9} S_{ro}' r \left[\frac{1.3 + 0.7 \dfrac{r^2}{R^2}}{1 - \dfrac{r^2}{R^2}} \right] \quad \cdots \cdots \cdots \quad [51a]$$

38   It is evident from Equations [44] to [51] that the additional stresses and radial elongation due to internal and external radial loads applied simultaneously are equal in value to the sum of the corresponding quantities when the loads are applied separately. Hence, by using the curves in Figs. 14, 15, 18 and 19, the total additional stresses and elongation can be determined for a hollow cylindrical rotor with any internal and external radial loads. If the rotor given in the problem of Case II has an internal and external radial load of 2000 lb. per sq. in., the total values, stresses and elongation from Cases III and V, are —

15040 + 6260 + 4250 = 25550 lb. per sq. in. tangential stress at the internal radius
 8000 + 4260 + 2250 = 14510 lb. per sq. in. tangential stress at the external radius
 1120 + 1366 − 630 = 1850 lb. per sq. in. radial stress at 0.8 radius
0.005536 + 0.00254 + 0.00158 = 0.00966 in. elongation in the external radius.
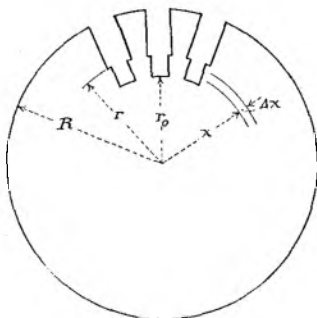

Fig. 21

39   *Case VII.   Calculation of Stresses in Rotor Teeth* (Fig. 21). — In this problem it is assumed that the equivalent weight per cubic inch of the materials in the slots is the same as that of the iron and that the end turns are supported independently.

40   Let r = radius at base of slots
       R = radius at top of slots
       w = angular velocity in radians per sec.
       W = weight per cu. in. of material
       p = ratio of tooth width at the bottom of the slots to the circumference at this radius
       S_r = radial stress at the root of the teeth

Consider an annular ring at a distance x from the center and radial depth Δx. Then —

$\Delta V = Kx \cdot \Delta x$ = volume of elementary ring

$\Delta M = K\dfrac{W}{g} x \cdot \Delta x$ = mass of elementary section

$\Delta F = \dfrac{W}{12g} Kx^2 w^2 \cdot \Delta x$ = centrifugal force acting on ring

$F = \dfrac{W}{12g} Kw^2 \displaystyle\int_r^R x^2 \, dx$ = total centrifugal force of material between radii r and R

$= \dfrac{W}{12g} Kw^2 \dfrac{R^3 - r^3}{3}$

41 This total force F must be carried by the iron section at the root of the teeth.

Krp = iron section at the root of the teeth, and

$rpKS_r$ = total force acting on the material at the root of the teeth

$rpKS_r = \dfrac{W}{12g} Kw^2 \dfrac{R^3 - r^3}{3}$

$pS_r = \dfrac{W}{12g} \dfrac{w^2}{3r} (R^3 - r^3)$

$= \dfrac{W}{36g} w^2 \dfrac{R^3}{r} \left(1 - \dfrac{r^3}{R^3} \right)$

$= \dfrac{4W}{g} V_{fr}^2 \left(\dfrac{R}{r} - \dfrac{r^2}{R^2} \right)$

42 If the maximum permissible stress at the root of the teeth is S_r', then —

$$p = \frac{0.0354}{S_r'} V_{fr}^2 \left[\frac{R}{r} - \frac{r^2}{R^2} \right] \quad \cdots \cdots \quad [52]$$

or for a given tooth section at the base of the teeth,

$$S_r = \frac{0.0354}{p} V_{fr}^2 \left[\frac{R}{r} - \frac{r^2}{R^2} \right] \quad \cdots \cdots \quad [53]$$

NUMERICAL EXAMPLE:

If $p = 0.5$, $\dfrac{r}{R} = 0.8$, and $V_{fr} = 400$ ft. per sec., then —

$$S_r = \frac{0.0354}{0.5} \times (400)^2 (1.25 - 0.64)$$

$$= 6900 \text{ lb. per sq. in. radial stress at the root of the teeth.}$$

43 *Case VIII. Stresses in Teeth at Bottom of Air Ducts* (Fig. 22). — In the rotors of radial-slot turbo-alternators air channels are often provided at the bottom of the slots for cooling purposes. The iron section at the bottom of the channels must carry the total centrifugal force due to all of the external material.

44 Let R = external radius of the rotor

r = radius at the bottom of main slots

r_o = radius at the bottom of the air channel

p_1 = ratio of the tooth width at the bottom of air channels to the circumference at this point

S_r = lb. per sq. in. stress in iron at the bottom of the teeth.

Then—

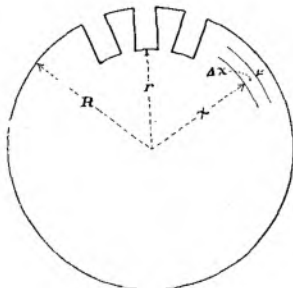$$K(1 - p_1)\, r_o = \text{total width of the air channels}$$

FIG. 22

and—

$$K[x - (1 - p_1)\, r_o] = \text{total iron width at any section}$$

$$\Delta V = K\,[x - (1 - p_1)\, r_o]\,\Delta x = \text{volume of elementary section}$$

$$\Delta M = K\,\frac{W}{g}[x - (1 - p_1)r_o]\,\Delta x = \text{mass of elementary section}$$

$$\Delta F_o = \Delta M \cdot \frac{x}{12}\, w^2 = \text{centrifugal force acting on elementary section}$$

$$= K\,\frac{W}{12g}\, w^2\,[x^2 - (1 - p_1)r_o x]\,\Delta x$$

$$F_o = K\,\frac{W}{12g}\, w^2 \int_{r_o}^{r}\left[x^2 - (1 - p_1)\, r_o x\right]dx$$

$$= K \frac{W}{12g} w^2 \left[\frac{x^3 - r_o{}^3}{3} - \frac{1 - p_1}{2} (r^2 - r_o{}^2) \right]$$

45   The total centrifugal force acting on the section at the bottom of the air channel is $F + F_o$, where F is determined as in the previous case.

$$F + F_o = K \frac{W}{12g} w^2 \left[\frac{R^3 - r_o{}^3}{3} + \frac{1 - p_1}{2} (r_o{}^3 - r_o r^2) \right]$$

But $r_o p_1 K S_r$ = total force acting on the teeth at the bottom of the air channel, therefore —

$$r_o p_1 K S_r = \frac{W}{12g} K w^2 \left[\frac{R^3 - r_o{}^3}{3} + \frac{1 - p_1}{2} (r_o{}^3 - r_o r^2) \right]$$

$$p_1 S_r = \frac{W w^2}{12 g r_o} \left[\frac{R^3 - r_o{}^3}{3} + \frac{r_o{}^3 - r_o r^2}{2} \right] - \frac{W w^2 p_1}{12 g r_o{}^2} \left[r_o{}^3 - r_o r^2 \right]$$

$$p_1 \left[S_r - \frac{W w^2}{12 g r_o{}^2} (r_o r^2 - r_o{}^3) \right] = \frac{W w^2}{12 g r_o} \left[\frac{R^3}{3} + \frac{r_o{}^3}{6} - \frac{r_o r^2}{2} \right]$$

$$S_r = \frac{0.1061}{p_1} V_{fr}{}^2 \left[\frac{R}{3 r_o} + \frac{r_o{}^2}{6R^2} - \frac{r^2}{2R^2} - \frac{p_1}{2} \left(\frac{r_o{}^2}{R^2} - \frac{r^2}{R^2} \right) \right] \quad \cdot \quad \cdot \quad [54]$$

or —

$$p_1 = \frac{0.1061 V_{fr}{}^2 \left[\dfrac{R}{3 r_o} + \dfrac{r_o{}^2}{6R^2} - \dfrac{r^2}{2R^2} \right]}{S_r + 0.0531 V_{fr}{}^2 \left[\dfrac{r_o{}^2}{R^2} - \dfrac{r^2}{R^2} \right]} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad [55]$$

NUMERICAL EXAMPLE:

If $p_1 = 0.50$, $V_{fr} = 400$ ft. per sec., $\frac{r}{R} = 0.8$, and $\frac{r_o}{R} = 0.6$, then —

$$S_r = 0.1061 \times \frac{(400)^2}{0.5} \left[\frac{1}{3 \times 0.6} + \frac{(0.6)}{6} - \frac{(0.8)^2}{2} - \frac{0.5}{2} (0.6^2 - 0.8^2) \right]$$

$$= 12,400 \text{ lb. per sq. in. at the bottom of the air channels.}$$

DISCUSSION

G. M. EATON.   This paper places a very valuable set of tools in the hands of designers of apparatus for high rotative speed. The text is very clear in its presentation of the fundamental principles, and the equations as finally derived are in practical and usable form.

In common with all formulas derived by pure analytic methods

the application in actual designs must be guided by a full knowl-
edge of the materials to be employed, and the conditions to be
met in service.

This may be well illustrated by one of the problems facing
the designer of a rotor that is to be made from a single forging. On
the face of the formulas there is no possible argument for an axial
hole through the rotor, as the stresses at the surface of the hole
are far higher than those existing at the axis in a similar rotor
with a solid center. But after all, we are more interested in the factor
of safety than we are in stress, and the question forces itself upon
us, whether in this large forging the center actually is solid. The
steel mills tell us that even when the cropping of the ingot has
removed all appearance of piping, we must expect an area of weak
structure extending for an indeterminate distance below the pipe.
Furthermore, comparatively small temperature inequalities over
the length of the ingot combine with the unavoidable temperature
gradients normal to all radiating surfaces, to set up severe and
sometimes destructive tensile stresses in the last metal to cool;
namely, the axial core. The steel mills, therefore, are practically
unanimous in recommending an axial inspection hole throughout
the entire length of large forgings for high-speed rotors.

Apparently then, we must choose between a comparatively
low calculated stress with quite indeterminate material on the
one hand, and a high stress with material whose characteristics are
in a measure capable of determination. This latter stress we may
find leaves a factor of safety which appears at first thought to be
lower than we would like to provide, when compared with the
known elastic and endurance limits of the material employed.

From the standpoint of endurance limit, it is usually com-
forting to realize the infrequency with which the stress cycle from
zero speed to full speed is completed by large high-speed rotating
machinery.

There is, however, a further consideration in connection with
normal speed stresses which constitutes a strong argument in favor
of providing the axial hole. If the stress under discussion is close
to the elastic limit at normal speed, it is entirely possible that under
the test or occasional service over-speed condition, the rapidly
mounting stress will pass the elastic limit. This is true whether a
hole is intentionally provided, or whether an incipient weakness or
rupture exists in a supposedly solid rotor. In the former case the
material at the smooth surface of the hole will take a fairly uniform

permanent set, while in the latter a running crack may possibly develop.

Analyzing the effect of plastic flow of the metal adjacent to the hole, we find that since there can be no reduction of the resultants of centrifugal action at any given speed, and since the material undergoing plastic flow can offer little more than its elastic resistance to these resultants, the stress must be shared by material other than that undergoing plastic flow. Obviously, a loading back occurs and the less strained material adjacent to that which first flows will take on added stress. Some of this material will also enter the stage of plastic flow, the amplitude of such flow gradually decreasing with increase of radius and this increase being with no sharp line of demarcation, until a region is reached where the stress is exactly equal to the elastic limit. Outside of this region all phenomena remain elastic.

When the period of over-speed is passed and the speed begins to drop, the material that has been stretched beyond the elastic limit is too long to reassume the identical relations that it previously bore to adjacent material, and it enjoys a relief from the stress it originally endured at normal speed, this relief being approximated by the distance of plastic flow per unit of length as measured on the elastic scale of the material.

· Therefore, a machine which has successfully passed an over-speed test will frequently run at normal speed with a lower maximum stress than occurred before the machine was over-speeded, and therefore with a greater elastic factor of safety than would appear from the formulas given in the paper.

We have referred to elastic factor of safety. This term is applied to the ratio of the elastic limit to the working stress. But in the case under consideration this ratio is quite misleading, and the actual factor of safety may be many times as great as this ratio would make it appear.

When we pull a standard test piece, we find that our material will undergo a certain per cent of elongation between gage points before it breaks. This elongation is not evenly distributed over the distance between gage points and is therefore not a fully representative characteristic of the material, but it is, nevertheless, somewhat indicative. The reduction of area is a much more definite description of the material.

We can pull a large number of test pieces and we find quite consistent results. Now if we base our factor of safety on the

elongation or reduction of area of the metal undergoing plastic flow, we have a more truly descriptive factor of safety which we may term the plastic factor of safety. The test piece will never break with average material till a certain reduction has occurred, and if the movement is positively stopped at a certain per cent of the available ultimate reduction, the ratio of the ultimate to the actual reduction is a logical safety factor. Numerically, we would never accept as low a plastic factor as the usual elastic factor of safety, but as a real and usable factor, the plastic factor of safety must be accepted in cases of the type under discussion.

The fundamental point of difference between the usual elastic beam condition and the setting of combined elastic and plastic phenomena is that in the former case the material follows the force application without relief of stress, under the laws of elasticity, whereas in the latter there occurs, as stated, a loading back of stress to lower stressed parts so that more and more material is loaded as the material in the critical region refuses to assume the share of the load it previously bore.

There is therefore no cause for anxiety in cases of combined elastic and plastic flow over what appears to be a low elastic factor of safety when further investigation along the lines suggested reveals an ample plastic factor of safety.

WILLIAM KNIGHT. The remarks made by Mr. Laffoon about the desirability of having a more comprehensive analysis of stresses in rotating elements of electrical machines than is given in hand-books are quite to the point.

As far, however, as stresses in rotating disks with a hole at the center are concerned, the writer believes that the analysis of such stresses, as it is given in Stodola's book, The Steam Turbine, is quite complete and comprehensive.

The only objection to the formulas given by Stodola, in my estimation, is that they involve an elaborate mathematical manipulation of numerical values and too many chances of making errors.

To eliminate, as far as possible, chances of errors in the calculation of stresses in rotating disks with a hole at the center and to make such calculation accessible to any ordinary draftsman, the writer has simplified the formulas given by Stodola and has expressed them in such a form as to make it possible to substitute the use of graphical methods for the long calculations involved in the use of them.

The two general equations given by Stodola for disks are as follows:

$$\sigma_r = \frac{E}{1 - v^2} \left[(3 + v)kx^2 + (\psi_1 + v)b_1 x^{\psi_1 - 1} + (\psi_2 + v)b_2 x^{\psi_2 - 1} \right] \quad . \quad . \quad [\mathbf{56}]$$

$$\sigma_t = \frac{E}{1 - v^2} \left[(1 + 3v)kx^2 + (1 + \psi_1 v)b_1 x^{\psi_1 - 1} + (1 + \psi_2 v)b_2 x^{\psi_2 - 1} \right] \quad [\mathbf{57}]$$

and refer to disks with a hole in the center having a profile represented by the equation:

$$y = \frac{c}{x^a}$$

where y is the thickness of the disk at any radius x, c is a constant, and a may be either positive or negative or equal zero.

For $a = 0$ we have the case of a disk of uniform thickness.

For positive values of a we have the disk section used by de Laval for small wheels.

For negative values of a we have a profile of disk increasing in thickness from the bore toward the periphery.

The exponent a is given by:

$$a = \frac{\log \frac{y_1}{y_2}}{\log \frac{x_2}{x_1}}$$

where y_1 and y_2 are the thicknesses of the disk at the corresponding radii x_1 and x_2.

b_1 and b_2 in the above formulas are two constants depending on the outline of the disk and on the presence or absence of other forces acting on the disk besides the centrifugal force of its own mass.

$$\psi_1 = \frac{a}{2} + \sqrt{\frac{a^2}{4} + av + 1}$$

$$\psi_2 = \frac{a}{2} - \sqrt{\frac{a^2}{4} + av + 1}$$

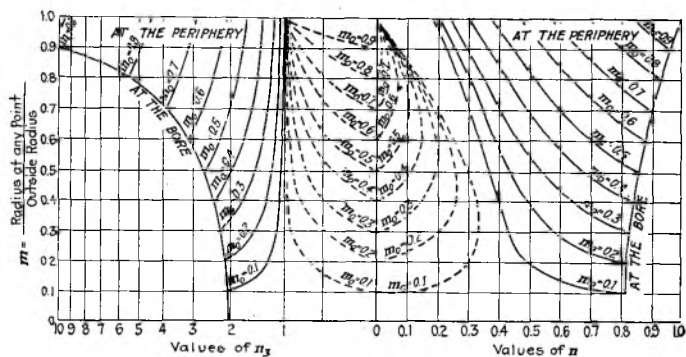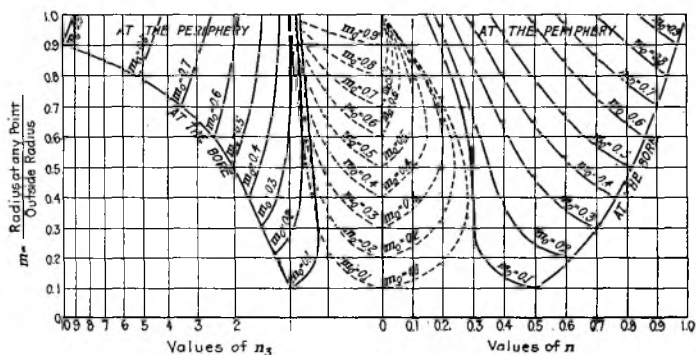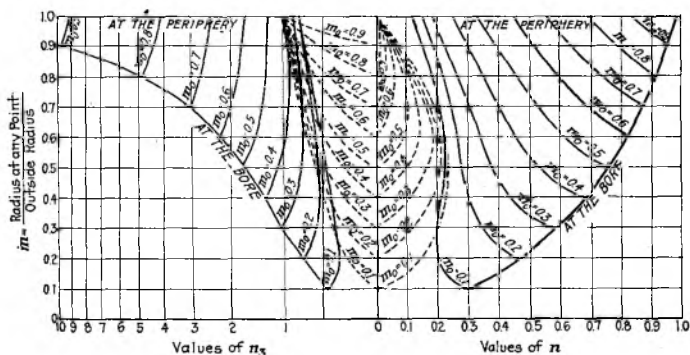$$v = \text{Poisson's ratio} \qquad \text{(for steel } v = 0.3)$$

and, with the inch-lb-second system:

ω = angular velocity in radians per sec.

μ = unit of mass of metal of disk (steel = 000,725)

E = modulus of elasticity (steel = 30,000,000)

$$k = \frac{-(1 - v^2) \mu \omega^2}{E \left[8 - (3 + v)a \right]}$$

FIG. 23   STEEL DISK OF UNIFORM SECTION ($\alpha = 0$) WITH A CENTRAL HOLE

FIG. 24   STEEL DISK OF HYPERBOLIC SECTION ($\alpha = 0.5$) WITH A CENTRAL HOLE

FIG. 25   STEEL DISK OF HYPERBOLIC SECTION ($\alpha = 1$) WITH A CENTRAL HOLE
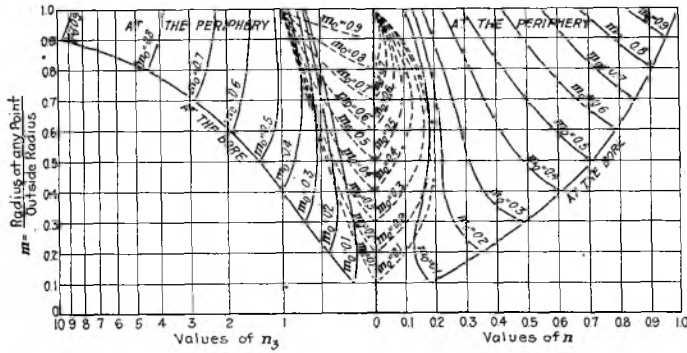
FIGS. 23 TO 25   VALUES OF n

FIG. 26   STEEL DISK OF HYPERBOLIC SECTION ($\alpha = 1.5$) WITH A
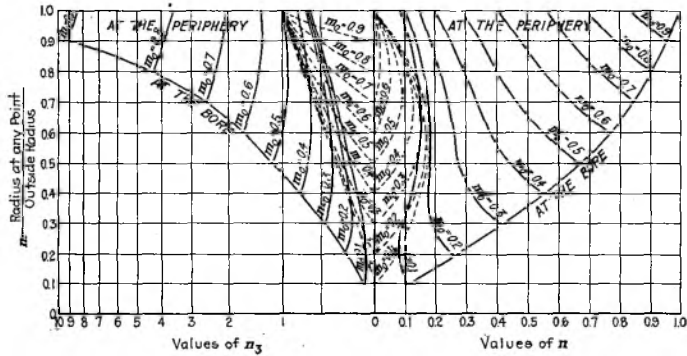CENTRAL HOLE

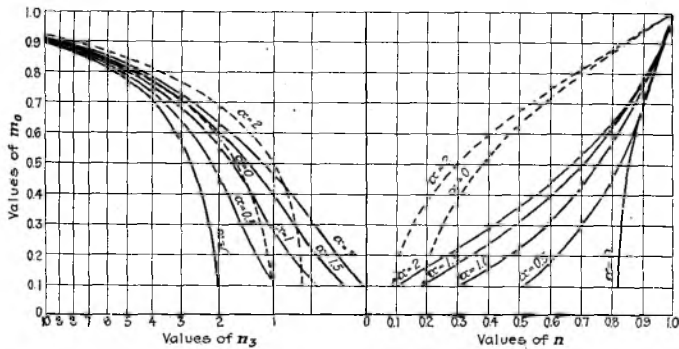FIG. 27   STEEL DISK OF HYPERBOLIC SECTION ($\alpha = 2$) WITH A
CENTRAL HOLE

FIG. 28   STEEL DISK OF HYPERBOLIC SECTION

FIGS. 26 TO 28   VALUES OF n

x = radius at any point in inches.

In the simplified form in which Stodola's formulas are expressed, the following. symbols will be used:

σ_t, σ_r = tangential and radial stresses at any radius due to the centrifugal forces of the disk only

σ_t^1, σ_r^1 = tangential and radial stresses at any radius due to the centrifugal force of the projecting masses only

σ = tangential stress in a thin ring of the same material as the disk, and rotating at a speed equal to the peripheral speed of the disk

m = ratio of radius at any point to the outside radius

m_0 = ratio of radius at the bore to the outside radius.

By introducing the two terms m and m_0, Equations [56] and [57] can be expressed as follows:

$$\sigma_r = \sigma \left\{ A \left[- m^2 + \frac{m^{\psi_1 - \psi_2}(m_0^{3-\psi_2} - 1) - (m_0^{3-\psi_2} - m_0^{\psi_1 - \psi_2})}{m^{1-\psi_2}(m_0^{\psi_1 - \psi_2} - 1)} \right] \right\} = \sigma n \quad [58]$$

$$\sigma_t = \sigma \left\{ B \left[- m^2 + \frac{C m^{\psi_1 - \psi_2}(m_0^{3-\psi_2} - 1) + D(m_0^{3-\psi_2} - m_0^{\psi_1 - \psi_2})}{m^{1-\psi_2}(m_0^{\psi_1 - \psi_2} - 1)} \right] \right\} = \sigma n \quad [59]$$

The values of n in Equation [58] are obtained on the right hand side of the diagrams, Figs. 23 to 28, from dotted line curves for corresponding values of m and m_0.

The values of n in Equation [59] are also obtained on the right hand side of the same diagrams, but they are given by the full line curves there shown.

The constants A, B, C, D used in Equations [58] and [59] are:

$$A = (2 - v^2)(1 - v^2)B$$

$$B = \frac{2 - v^2}{8 - (3 + v)a}$$

$$C = -(2 - v^2)(1 - v^2)\psi_2$$

$$D = (2 - v^2)(1 - v^2)\psi_1$$

which for $v = 0.3$ become:

$$A = 1.74B$$

$$B = \frac{1.91}{8 - 3.3\alpha}$$

$$C = -1.74\psi_2$$

$$D = 1.74\psi_1$$

Similarly, introducing again the two terms m and m_0, Equations [56] and [57] can be expressed in this simplified form:

$$\sigma_r{}^1 = \sigma_3 \frac{m_0{}^{\psi_1-\psi_2} - m^{\psi_1-\psi_2}}{m^{1-\psi_2}(m_0{}^{\psi_1-\psi_2} - 1)} = \sigma_3 n_3 \quad \ldots \quad [60]$$

$$\sigma_t{}^1 = \sigma_3 \frac{\psi_1 m_0{}^{\psi_1-\psi_2} - \psi_2 m^{\psi_1-\psi_2}}{m^{1-\psi_2}(1 - m_0{}^{\psi_1-\psi_2})} = \sigma_3 n_3 \quad \ldots \quad [61]$$

where $\sigma_3 =$ centrifugal force of projecting masses per square inch of surface at the periphery of the disk.

Values of n_3 in Equations [60] and [61] are obtained on the left hand side of diagrams Figs. 23 to 28, from dotted line curves for Equation [60], and from full line curves for Equation [61].

The usefulness of these diagrams is quite evident. In order to find the radial and tangential stresses at any point of a disk with a hole at the center, either of a uniform section (same as it is in the case considered by Mr. Laffoon) or of a hyperbolic section (which includes the case of the disk of uniform section), same as considered by Stodola, all we need to know is the value of α (which in the case considered by Mr. Laffoon is equal to zero), the values of m and m_0 and calculate on the slide rule the values of σ and σ_3. The diagrams will give us at a glance the values of radial and tangential stresses at any point between the bore and the periphery of the disk.

Diagrams Figs. 23 to 27 have been calculated for values of α varying from $\alpha = 0$ to $\alpha = 2$ ($\alpha = 0$, $\alpha = 0.5$, $\alpha = 1.0$, $\alpha = 1.5$, $\alpha = 2$) and for values of m and m_0 varying from 0.1 to 1.0. The diagram shown in Fig. 28 gives the values of tangential stress only at the bore and at the periphery of the disk for corresponding values of α and m_0. This diagram will be found to be particularly useful in determining the influence of a change in the design of the disk over its strength.

In order to allow anybody to plot these diagrams, the values of $\frac{\sigma_r}{\sigma}$ corresponding to the values of n in Equation [58] may be

taken from Tables 6 to 10, p. 517, *Machinery*, February 1918, which I published some years ago.

Tables 1 to 5 on the same page give the values of $\frac{\sigma_t}{\sigma}$ corresponding to the values of n in Equation [**59**].

Tables 16 to 20 on p. 518, February 1918 issue of *Machinery*, give the values of $\frac{\sigma_r^1}{\sigma_3}$ corresponding to the values of n_3 in Equation [**60**].

Tables 11 to 15 on the same page give the values of $\frac{\sigma_t^1}{\sigma_3}$ corresponding to the values of n_3 in Equation [**61**].

Figs. 23 to 28 were published by the writer in *Engineering* (London), August 3, 1917.

The particular case of the design of the rotor core of an electrical machine, considered by Mr. Laffoon in his paper, the writer has given in the *Electrical World*, January 12, 1918.

A. L. KIMBALL. The subject of stress around a hole has been investigated by an interesting method. A rubber diaphragm was ruled with squares, and by means of the distortion of the squares around the hole, an approximate value of the stress produced at the hole was arrived at. The stress did not depend on the size of the hole. The rubber must not be strained too much, or an error will result.

In Dr. Coker's laboratory in London some stressed celluloid sheets with elliptical holes were studied photo-elastically. It was found that the maximum stress increased as the ellipse was elongated and the amount of the stress was found to check almost exactly with the theoretical value. Compression at the top and bottom of the ellipse was discovered as the theory predicted. This can be used to show what occurs at a crack or notch, which may be considered a very elongated ellipse.

PAUL HEYMANS. The mathematical treatment of problems of the kind which have been considered in Mr. Laffoon's paper can only be carried out at the price of simplifying assumptions. How much these assumptions cause the calculated stresses to depart from the actual stresses which will arise in the structure is always an uncertain question. The factor of safety has to cover this factor of doubt. We venture to say that the photo-elastic method opens a promising possibility for the investigation of the majority of

these problems. In this method the stresses are directly measured. The only restricting assumptions are that the material is isotropic and that it obeys Hooke's law of linear proportionality between stress and strain.

The stress analysis in the photo-elastic method is made on celluloid models. For those materials which have beyond the elastic limit a stress-strain curve similar to that of celluloid, the results obtained above that limit, where the material no more obeys Hooke's law but follows some other stress-strain curve, still hold. The stress-strain curve of celluloid is similar to the stress-strain curve of mild steel. Moreover, it can be acted upon by proper nitration of the nitro-cellulose entering in the compositions of celluloid. Therefore, a model of proper celluloid analysed below and beyond the elastic limit should throw fuller light on the subject treated in this paper.

The question of the stresses set up by a circular hole drilled in the center of a rotating disk has been raised. An Austrian physicist, Dr. Leon,[1] has presented a complete analytical treatment of the subject of stresses around a hole, and has shown that in the case of a plate uniformly stressed in all directions, a hole drilled in the plate will present around the boundary a stress double the previous uniform stress. In the case of an infinite plate uniformly stressed jn one direction, this same discontinuity will set up stresses equal to three times the mean stress.

We have in our photo-elastic laboratory subjected a disk of celluloid to a uniform stress, and re-examined it after drilling in its center a small hole. The values of the stresses determined confirmed Dr. Leon's analytical results.

S. A. Moss. A rotor with a hole in the middle is certainly stressed much more than it would be if there were no hole. The radial stresses on either side of the hole do not balance each other as they do where there is no hole. A simple illustration is that of the effect produced by sticking a pin through a thin sheet of stretched rubber. The pin hole instantly enlarges. The conditions are the same at the center of the wheel.

ALPHONSE A. ADLER said that everyone who analyzed such a subject as this commenced with equations of the maximum strain theory. He would like some one to base the equation on maximum shear

[1] Dr. A. Leon, Oesterreichische Wochenschrift für den öffenlichen Baudienst Wien, Februar 1908, pp. 163–168.

theory since this has recently become very prominent.   He thought that the photo-elastic method would be valuable in determining whether to use the maximum strain or the maximum shear theory.

The author follows the usual custom of assuming that in the case of rotor teeth the stress in the main body of the rotor is only affected by the increased loading due to the teeth.   From the paper by Heymans and Kimball[1] on Railway Motor Pinions it is shown how seriously the presence of teeth on the gear affects the distribution of stress in the hub.   Perhaps this same influence is present in the case of armatures with toothed cores.   A photo-elastic experiment would do much to indicate the departure from the simple assumptions given in this paper.

[1] See paper No. 1859.