Influence of Lashing and Centrifugal Force on Turbine-Blade Stresses

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This paper gives a stress analysis for rotating segments of lashed blades under a given load. By using difference calculus, results were obtained, which apply to any blade out of any number of blades.

OWING to steam forces and to centrifugal force, the turbine blades are often highly stressed; moreover, they may be subject to vibration. In order to relieve these blades, it has become customary to connect them in groups with lashing wires, shrouding strips, or similar reinforcements. These constructions stiffen the blades, and also provide damping in case of vibration.

This paper is intended as a first attempt to investigate the behavior of such rotating groups of lashed blades.

The steam load and the centrifugal force on the blades may be assumed to be known, but the magnitude of the vibration stresses (which can be predominant) depends on the impulses and the damping and can be determined only by tests.

We will limit ourselves here by assuming definite forces, equal on all the blades of the group. It will be shown that the effect of these forces is not at all equal on the individual blades of the segment; the stresses vary in a characteristic oscillating manner.

The difference calculus,[2] which was developed to study groups of similar objects and which has proved to be extremely useful in dealing with civil-engineering problems such as bars on many supports, built-up beams, bridges, platforms on columns, etc., can also be successfully applied to the case of lashed blades.

Before going into detail analysis, it might be worth while to give a summarized physical description of the results obtained in this paper. For simplicity we have taken blades of constant cross-section connected with one lashing wire at the tip. Furthermore, we assumed the load and the wire to be in the same plane as the maximum axis of inertia of cross-section of the blades. Without any mathematics, it can immediately be seen that the load, which is equally divided over all the blades of the group, will not impose the same stresses on all the blades; the two end blades are only supported by the lashing wire on one side of the blades; all the other blades are supported on both sides.

Assume, for example, a very flexible lashing wire. Then the deflection of all the blades, and especially the angle of rotation at the tip, will be practically the same for all blades. Therefore the stresses in the wire will be the same in all the sections. This, however, means that the bending moment at the tip of the end blades is only half as great as the moment at the tip of all the other blades.

The opposite effect enters for an extremely thick wire. Here the angle of rotation at all the blade tips must necessarily be zero, the bending moments at the tips of all blades are equal, but the lashing wire is much more stressed just beside the end blades than in any other place.

Using difference calculus, it is possible to find the deflection and the bending moments for any blade out of a group of any number of blades of any dimensions.

Results for a segment of six blades under stationary lateral load are shown in Figs. 6, 7, and 18*a*. The bending moments are plotted as a function of a dimensionless quantity λ. In this way these curves obtain universal value for all the dimensions of blade and wire which we might wish to choose (for notation see list at end of paper). In order to find the value of any bending moment shown in these figures, it is necessary to multiply the numerical value, given on the ordinate axis, with $I_bE_b\varphi_0/l$. As φ_0 is the angle of rotation which would exist at the blade tip for the same loading without lashing wire, $I_bE_b\varphi_0/l$ can be looked upon as a "fundamental bending moment."

The maximum stress reduction which can be achieved by a single lashing wire is 33.3 per cent for uniformly distributed load and 50 per cent for a load concentrated at the tip.

The centrifugal force, apart from producing tensile stresses in the blades, plays an important role in reducing the bending stresses set up by the steam load as it tends to bend the blades back in their neutral position. A differential equation can be derived for the general case of blades with variable cross-section. Correction factors, taking into account the effect of the centrifugal force on single blades with various kinds of loading, may be found in Figs. 9, 10, and 11. They are given in function of k, which is a dimensionless parameter, proportional with the centrifugal force (see list of notations). The effect of the centrifugal force on bending stresses in long tapered low-pressure blades is enormous; it was found that a bending moment 1, applied at the tip of a 40-in. blade, produces a bending moment of only $^1/_{30}$ at the base.

The effect of centrifugal force on laterally loaded blades, lashed together in a segment, is shown in Figs. 12, 13, 14, and 18*b*. The maximum values of the bending moments in the group are given.

Another problem dealt with is the effect of the centrifugal force on the wire (no load on blades). The resulting forces and bending moments are shown in Figs. 15, 16, 17, and 18*c*. These values can again be slightly corrected by taking into account the centrifugal force acting on the blades.

The final picture of the stress distribution over a group of rotary blades under lateral load is obtained by superposition; a specific example is shown in Fig. 18*d*.

1—Lashing on Stationary Blades With Lateral Load

As a first problem, let us investigate the influence of lashing on a group of stationary blades with steady load.

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[2] See Bleich-Melan, "Die Gewöhnlichen und partiellen Differenzengleichungen der Baustatik," Springer, Berlin, 1927.

Contributed by the Applied Mechanics Division and presented at the Semi-Annual Meeting, Chicago, Ill., June 26 to July 1, 1933, of The American Society of Mechanical Engineers.

(*a*) *Segment of Three Blades; No Centrifugal Force.* As the most elementary interesting case take a group of three blades, equally loaded by lateral forces. At the point of intersection of blade and lashing wire we will assume all possible moments and forces (Figs. 1 and 2).

Using the notation listed at the end of this paper, we have for symmetry reasons:

$$\varphi_1 = \varphi_3,\ M_1 = M_3,\ M_1'' = M_3',\ M_2'' = M_2',$$
$$S_1 = S_3,\ T_1 = -T_3,\ T_2 = 0$$

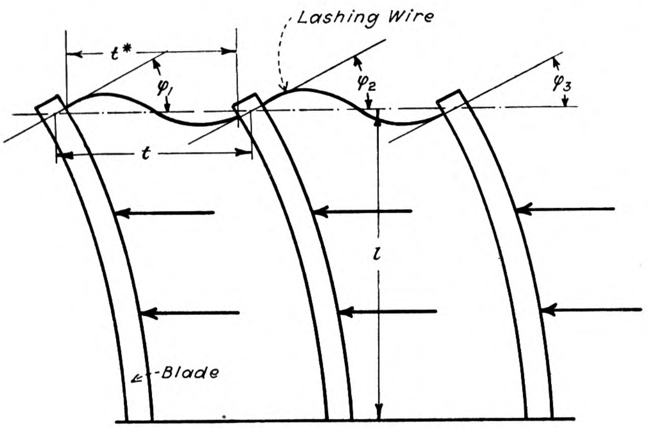

FIG. 1 THREE BLADES UNDER LATERAL LOAD

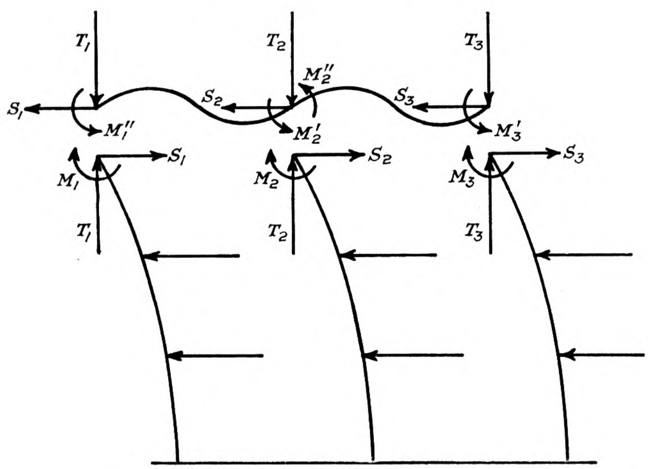

FIG. 2 REACTION FORCES ON LOADED SEGMENT

Furthermore, equilibrium conditions give:

$$M_1'' = M_1,\ M_3' = M_3,\ M_2' + M_2'' = M_2$$

$$S_1 + S_2 + S_3 = 0 \qquad -T_1 t^* = M_1 + (M_2/2)$$

This simplifies the picture of the reaction forces on the lashing wire to what is shown in Fig. 3.

The deformations can now be expressed as functions of the moments and forces. Tension, compression, and shear deformations are neglected. Using simple beam formulas,[3] we have for the lashing wire:

$$\frac{-M_1 t^{*2}}{6 I_s E_s} + \frac{M_2 t^{*2}}{6 I_s E_s} = t\varphi_2 \ldots\ldots\ldots\ldots\ [1]$$

$$\frac{-M_2 t^{*2}}{12 I_s E_s} + \frac{M_1 t^{*2}}{3 I_s E_s} = t\varphi_1 \ldots\ldots\ldots\ldots\ [2]$$

[3] See Timoshenko and Lessells, "Applied Elasticity," p. 74.

The angle of deflection which would exist at the blade tip for the same loading without a lashing wire will be designated as φ_0. Then, for the lashed group:

$$(M_1 + S_1 l/2)\frac{l}{I_b E_b} = \varphi_0 - \varphi_1 \ldots\ldots\ldots\ldots\ [3]$$

$$(M_2 - S_1 l)\frac{2}{I_b E_b} = \varphi_0 - \varphi_2 \ldots\ldots\ldots\ldots\ [4]$$

As the elongation of the lashing wire may be neglected compared with the deformations caused by bending, the tip deflection of each blade has to be the same:

$$\frac{M_1}{2} + \frac{S_1 l}{3} = \frac{M_2}{2} - \frac{2 S_1 l}{3} \ldots\ldots\ldots\ldots\ [5]$$

The Equations [1], [2], [3], [4], and [5] are sufficient to determine the quantities M_1, M_2, S_1, φ_1, and φ_2.

Putting $\lambda = \dfrac{t^{*2} I_b E_b}{t l I_s E_s}$, we find:

$$M_1 = \frac{6\lambda + 6}{\lambda^2 + 9\lambda + 6}\,\frac{I_b E_b \varphi_0}{l}; \qquad M_2 = \frac{12\lambda + 6}{\lambda^2 + 9\lambda + 6}\,\frac{I_b E_b \varphi_0}{l}$$

$$S_1 l = \frac{3\lambda}{\lambda^2 + 9\lambda + 6}\,\frac{E_b I_b \varphi_0}{l}; \qquad -S_2 l = \frac{6\lambda}{\lambda^2 + 9\lambda + 6}\,\frac{I_b E_b \varphi_0}{l}$$

$$\varphi_1 = \frac{\lambda(\lambda + 3/2)}{\lambda^2 + 9\lambda + 6}\varphi_0 \qquad \varphi_2 = \frac{\lambda^2}{\lambda^2 + 9\lambda + 6}\varphi_0$$

The factor $I_b E_b \varphi_0 / l$ is a bending moment, depending on the distribution of the load and on the length of the blade only.

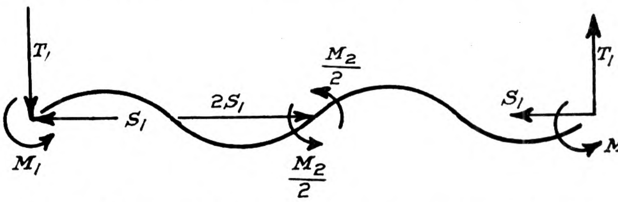

FIG. 3 GROUP OF THREE BLADES; REACTIONS ON LASHING

The λ is a dimensionless quantity, inherent with this problem. It takes into account the ratio of the flexibilities of blade and wire. By plotting the curves for moments, forces, tip angles, etc., as a function of λ, these curves obtain universal value for all the dimensions of blade and wire we might wish to choose. $\lambda = 0$ means a 100 per cent stiff wire; $\lambda = \infty$ represents an infinitely thin wire.

The stress distribution along the lashing wire does not depend on the way in which the blades are loaded (all quantities contain φ_0).

As Fig. 4 shows, the bending moment at the tip has for an infinitely stiff wire the same value for inner and outer blades; with a very thin wire the inner blade tip carries twice the stress of the outer ones. A 100 per cent stiff lashing wire gives at the outer blades a lashing-wire stress twice as high as at the inner blades. Decreasing the wire diameter within the practical range (up to $\lambda = 23.4$) increases the bending stress in the wire.

As can be seen from this simple example, there are for a group of three blades already 13 quantities involved—namely, two bending moments, two forces, and one angle at the tip of the middle blade, and one bending moment, two forces, and one angle at the tip of each end blade. By using conditions of symmetry and equilibrium, these 13 quantities could be reduced to 5, but still it is obvious that to analyze in this way a group of six or more blades with 28 or more variables is a somewhat burdensome task.

It is for work of this kind that the difference calculus has proved to be very effective. In contrast to differential calculus it studies the changes of functions for finite constant increments of the variables. Applied to our case, it makes it possible to obtain stresses and deflections for any blade out of a group of any number of blades.

(*b*) *Any Number of Blades; No Centrifugal Force.* Suppose we have n blades and consider the i^{th} blade and its neighbors (Fig. 5). Again using simple beam formulas, we find for the deformation of the lashing wire between the i^{th} and the $(i + 1)^{\text{th}}$ blade:

$$M_i'' = \frac{2I_sE_st}{t^{*2}}(2\varphi_i + \varphi_{i+1}) \ldots\ldots\ldots\ldots [1']$$

Also, between the i^{th} and the $(i - 1)^{\text{th}}$ blade:

$$M_i' = \frac{2I_sE_st}{t^{*2}}(2\varphi_i + \varphi_{i-1}) \ldots\ldots\ldots\ldots [2']$$

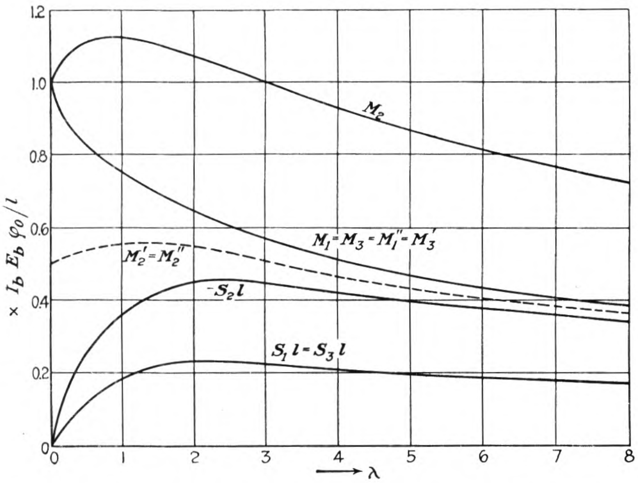

FIG. 4 BENDING MOMENTS AND SHEARING FORCES ON A SEGMENT OF THREE BLADES; LATERAL LOAD ONLY

The tip deflection of all blades is the same, and we can write:

$$\frac{S_il^3}{3I_bE_b} + \frac{M_il^2}{2I_bE_b} = Cl \ldots\ldots\ldots\ldots [5']$$

where C is a constant yet to be determined.

The tip angle of blade number i is then:

$$\varphi_i = \varphi_0 - \frac{M_il}{I_bE_b} - \frac{S_il^2}{2I_bE_b} = \varphi_0 - \frac{3}{2}C - \frac{M_il}{4I_bE_b} \ldots [3']$$

For equilibrium of the connecting point of blade and wire we have the condition:

$$-M_i' - M_i'' + M_i = 0 \ldots\ldots\ldots\ldots [4']$$

This set of equations leads to:

$$\varphi_{i-1} + (4 + 2\lambda)\varphi_i + \varphi_{i+1} = \lambda(2\varphi_0 - 3C) \ldots\ldots [6]$$

where λ is again $\frac{t^{*2}I_bE_b}{tlI_sE_s}$

Equation [6] establishes a relation between the values of the function φ at adjacent points with definite intervals; such a relation is called a "difference equation." There are many types of these equations, and for some of these the solutions are known. Equation [6] is a "linear" difference equation, and the solution can be proved to be:

$$\varphi_i = A\beta^i + \frac{B}{\beta^i} + \frac{\lambda(2\varphi_0 - 3C)}{6 + 2\lambda} \ldots\ldots\ldots\ldots [6']$$

where $\beta = -(\lambda + 2) + \sqrt{(\lambda^2 + 4\lambda + 3)}$ = small negative number. A and B are constants which have yet to be found.

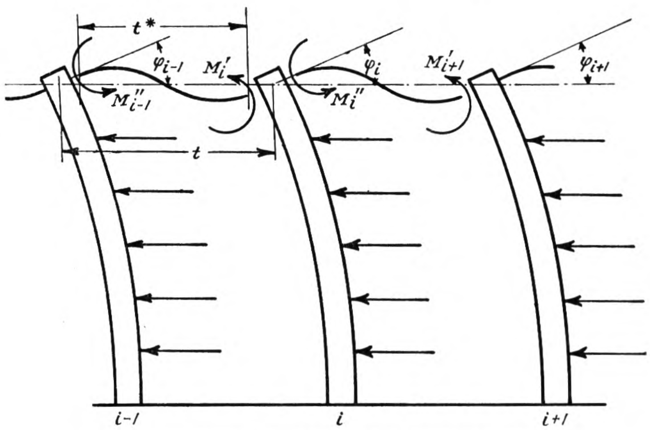

FIG. 5 THREE BLADES OF A LARGER GROUP

For symmetry reasons: $\varphi_i = \varphi_{n+1-i}$, so that [6′] becomes:

$$\varphi_i = A(\beta^i + \beta^{n+1-i}) + \frac{\lambda(2\varphi_0 - 3C)}{6 + 2\lambda}$$

The constants A and C are determined by the end condition

$$M_1' = 0 \therefore (2 + 2\lambda)\varphi_1 + \varphi_2 = \lambda(2\varphi_0 - 3C)$$

(compare [4′] and [6]), and by the condition of equilibrium for the wire:

$\sum_{i=1}^{i=n} S_i = 0$, which, in view of [5′] and [3′], can also be written:

$$\sum_{i=1}^{i=n}(2C - \varphi_0 + \varphi_i) = 0$$

Thus, after neglecting high powers of β we finally arrive at the formulas:

Tip angle of deflection:

$$\varphi_i = \frac{\varphi_0}{D}\left(\beta^{i-1} + \beta^{n-i} + \frac{2 + 2\lambda + \beta}{3}\right)$$

Tip bending moment:

$$M_i = \frac{4I_bE_b\varphi_0}{Dl}\left(-\beta^{i-1} - \beta^{n-i} + \frac{2 + 2\lambda + \beta}{\lambda}\right)$$

Bending moment in wire left of blade:

$$M_i' = \frac{2I_bE_b\varphi_0}{\lambda Dl}(2\beta^{i-1} + 2\beta^{n-i} + \beta^{i-2} + \beta^{n-i+1} + 2 + 2\lambda + \beta)$$

Bending moment in wire right of blade:

$$M_i'' = \frac{2I_bE_b\varphi_0}{\lambda Dl}(2\beta^{i-1} + 2\beta^{n-i} + \beta^i + \beta^{n-i-1} + 2 + 2\lambda + \beta)$$

Lateral force on tip:

$$S_1 = \frac{6I_bE_b\varphi_0}{Dl^2}\left(\beta^{i-1} + \beta^{n-i} - \frac{2}{n(1-\beta)}\right)$$

Bending moment at root:

$$M_i^* = M_0^* - \frac{2I_bE_b\varphi_0}{Dl}\left(\beta^{i-1} + \beta^{n-i} + \frac{4 + 4\lambda + 2\beta}{\lambda} - \frac{6}{n(1-\beta)}\right)$$

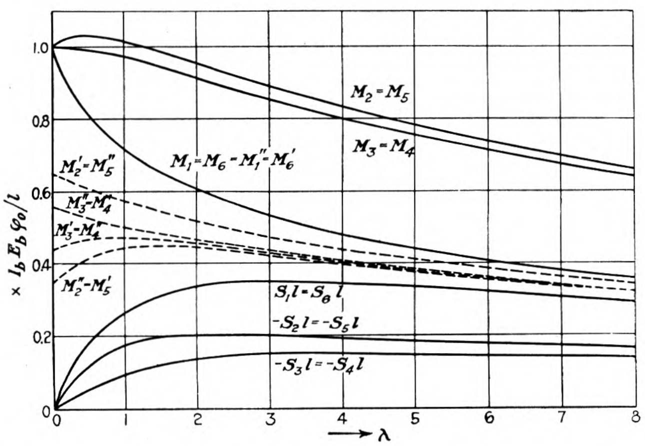

FIG. 6 BENDING MOMENTS AND FORCES ON A SEGMENT OF SIX BLADES; LATERAL LOAD ONLY

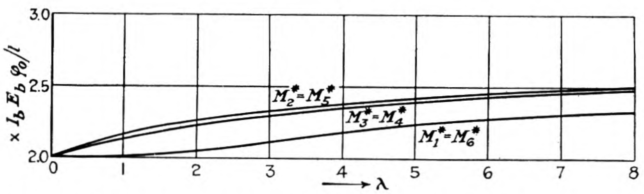

FIG. 7 ROOT BENDING MOMENTS FOR A GROUP OF SIX BLADES UNDER UNIFORMLY DISTRIBUTED LOAD

For notation see list at the end of this paper.

Calculations were made for a segment of six blades, and the results are shown in Figs. 6 and 7.

As may be concluded from the curves, the highest stress in the lashing wire is just beside the first and the last blade.

The second and the fifth blades have the greatest bending moment at the tip as well as at the root.

The weaker the lashing wire, the more variation between the values of the bending moments on the blades; the stronger the wire, the more difference between the values of the stresses in the wire sections. At the end blades the lashing wire and the blade tip carry the same bending moment. The bending stress at the root of the blade is for uniformly distributed load at least two times as high as the stress at the blade tip.

The maximum stress reduction which can be achieved by introducing a single lashing wire is 33.3 per cent for uniformly distributed load, and 50 per cent for a single force at each blade tip.

It was found that although the stresses in the individual blades vary a great deal, the deflections are almost identical for all the blades of the group. This suggested that under vibration the stress distribution around the lashing wire would be almost the same as under steady load.

E. Schwerin[4] studied the vibration of single lashed blades. A comparison between his calculations and these shows that the stress distribution during vibration (first mode) can be simulated by steady load so closely that the difference is only about 1 per cent. This is of great value for experimental work (Table 1).

TABLE 1 COMPARISON BETWEEN THE STRESSES SET UP DURING VIBRATION AND THOSE PRODUCED BY STEADY LOAD

(The table gives ratios between individual values and value for center blades $\lambda = 0.4$. n = total number of blades = 20. i = number of individual blade)

i	M vibrating	M steady	M' vibrating	M' steady	M'' vibrating	M'' steady
1	0.852	0.845	0	0	1.704	1.690
2	1.033	1.034	1.221	1.218	0.844	0.850
3	0.994	0.993	0.952	0.952	1.035	1.033
4	1.002	1.002	1.010	1.013	0.994	0.991
5	0.999	0.999	0.997	0.997	1.002	1.002
6–10	1	1	1	1	1	1

2—THE INFLUENCE OF CENTRIFUGAL FORCE ON BLADES WITH LATERAL LOAD

The centrifugal force reduces the deflection and the bending stress of radial beams with lateral load. Consider the case where a blade of variable cross-section but not "twisted" (the unbent axis of the blade being a radial line), is loaded by all kinds of bending forces, which are supposed to act in one of the principal planes of cross-section. When ξ is the variable distance from the tip, we may have bending moments M_ξ, lateral forces S_ξ, distributed load q_ξ, and centrifugal load p_ξ (per unit length). At a distance x from the tip, the variable moment of inertia will be called I_x and the deflection y_x (Fig. 8). The direction of the centrifugal force is taken parallel to the axis of the blade.

Assuming the simple beam theory to be correct, the bending moment at x will be:

$$I_xE_b\frac{d^2y_x}{dx^2} = \int_{\xi=0}^{\xi=x} q_\xi(x-\xi)d\xi - \int_{\xi=0}^{\xi=x} p_\xi(y_\xi - y_x)d\xi + \sum_{\xi=0}^{\xi=x}\left\{M_\xi + S_\xi(x-\xi)\right\}$$

Differentiating with respect to x, we get:

$$I_xE_b\frac{d^3y_x}{dx^3} + E_b\frac{d^2y_x}{dx^2}\frac{dI_x}{dx} = \int_{\xi=0}^{\xi=x} q_\xi d_\xi + \frac{dy_x}{dx}\int_{\xi=0}^{\xi=x} p_\xi\alpha\xi + \sum_{\xi=0}^{\xi=x} S_\xi$$

which is nothing else than the expression for the shearing force at x, consisting of a part

$$Q_x = \int_{\xi=0}^{\xi=x} q_\xi d\xi + \sum_{\xi=0}^{\xi=x} S_\xi$$

due to lateral load and a component

$$\frac{dy_x}{dx}P_x = \frac{dy_x}{dx}\int_{\xi=0}^{\xi=x} p_\xi d\xi$$

[4] *Zeitschrift für Technische Physik*, no. 8, 1927.

from the centrifugal force. We write:

$$\frac{dM}{dx} = I_x E_b \frac{d^3 y_x}{dx^3} + E_b \frac{d^2 y_x}{dx^2}\frac{dI_x}{dx} = Q_x + P_x \frac{dy_x}{dx} \quad \text{[7]}$$

or putting $\dfrac{dy_x}{dx} = v$, this becomes:

$$\frac{dM}{dx} = Q_x + P_x v \quad \text{[7a]}$$

while

$$M = I_x E_b \frac{dv}{dx} \quad \text{[7b]}$$

Numerical and graphical methods exist for the solution of these simultaneous equations. As they are linear, it follows that the principle of superposition may be applied; during rotation the stresses (deflections) caused by several lateral forces can be found by considering each force alone and superimposing the obtained stresses (deflections).

Application to Constant Section Blades. In this case $I_x = I_b =$ constant, and p_x will also be taken as a constant p. We will furthermore restrict ourselves to a uniformly distributed load q per unit length, a shearing force S, and a bending moment

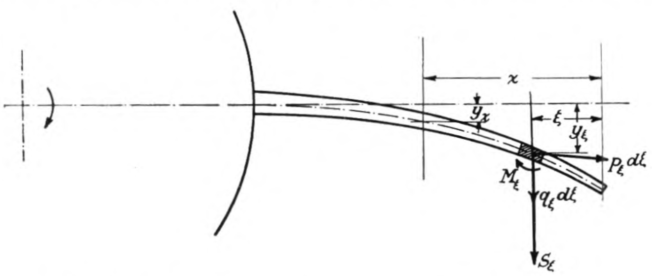

FIG. 8 INFLUENCE OF CENTRIFUGAL FORCE ON BLADE

M. S and M are acting at the blade tip. We have, then, for Equation [7]:

$$I_b E_b \frac{d^3 yx}{dx^3} = qx + S + px \frac{dy_x}{dx} \quad \text{[7']}$$

or when we call $q + p \dfrac{dy_x}{dx} = u$:

$$\frac{I_b E_b}{p}\frac{d^2 u}{dx^2} = ux + S \quad \text{[7'']}$$

Defining $\dfrac{I_b E_b}{p} = a^3$ and introducing $\dfrac{x}{a} = z$, this reduces to

$$\frac{d^2 u}{dz^2} = uz + \frac{S}{a} \quad \text{[8]}$$

The exact solution of [8] leads to a cylinder function of the $^1/_3$ order, but by means of a power series the equation can be easily solved. This is carried out in the Appendix.

As a result, ratios between the bending stress (deflection) during rotation and the stress (deflection) when standing still have been established. Fig. 9 shows these ratios for a distributed load, Fig. 10 shows curves for a bending moment at the tip, and Fig. 11 gives curves for a shearing force at the tip.

The ratios are plotted as function of a dimensionless variable $k = \dfrac{pl^3}{I_b E_b}$ and are universal for all blade dimensions.

In practise, values of k up to 3 and higher are common, and

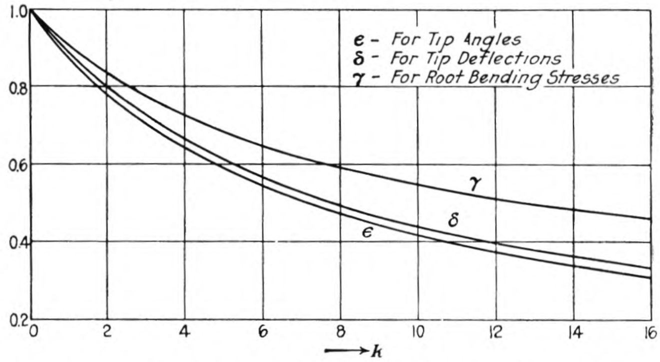

FIG. 9 CORRECTION FACTORS, TAKING INTO ACCOUNT THE EFFECT OF CENTRIFUGAL FORCE ON A BLADE WITH UNIFORMLY DISTRIBUTED LOAD

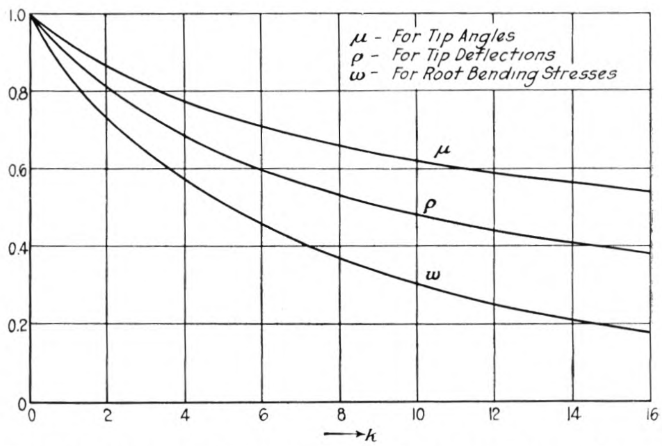

FIG. 10 CORRECTION FACTORS, TAKING INTO ACCOUNT THE EFFECT OF CENTRIFUGAL FORCE ON A BLADE WITH A BENDING MOMENT AT THE TIP

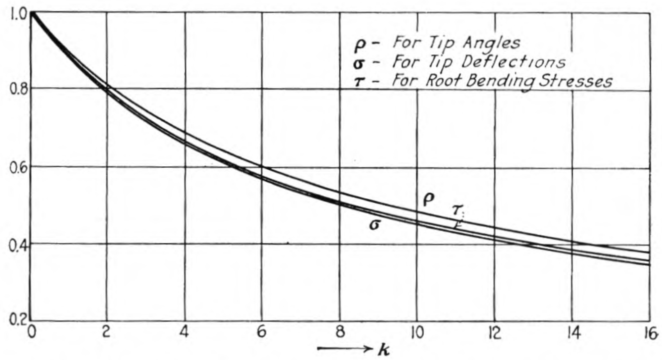

FIG. 11 CORRECTION FACTORS, TAKING INTO ACCOUNT THE EFFECT OF CENTRIFUGAL FORCE ON A BLADE WITH A LATERAL FORCE AT THE TIP

the reduction in bending stress may amount to 25 per cent or more for ordinary constant section blades.

3—LASHED BLADES UNDER BENDING LOAD; CENTRIFUGAL FORCE ON BLADES ONLY

Under Section 1 the influence of lashing on stationary blades was investigated, and under Section 2 we developed correction factors taking into account the effect of centrifugal force on blades with various kinds of leading. With these preliminaries, it is possible to compute stresses and deflections of centrifugally and laterally loaded lashed blades. Calculations were made for a

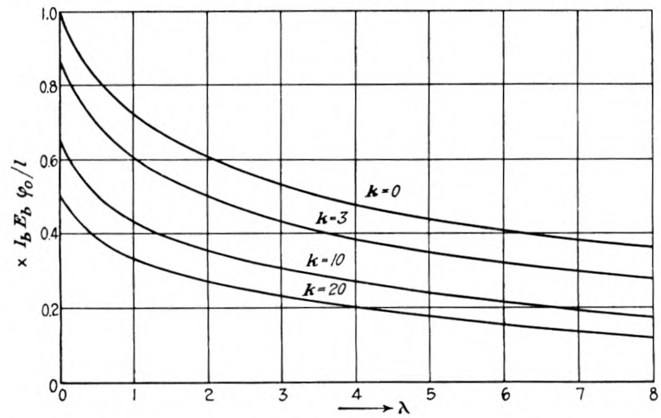

FIG. 12 GREATEST VALUE OF THE BENDING MOMENT IN THE LASHING WIRE FOR DIFFERENT VALUES OF THE CENTRIFUGAL FORCE (ACTING ON THE BLADE ONLY)

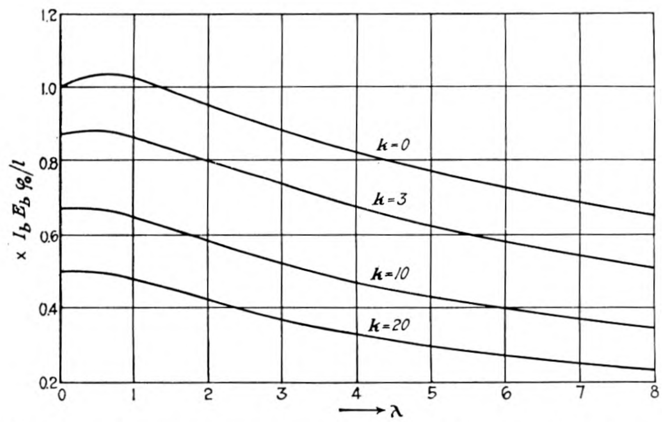

FIG. 13 MAXIMUM VALUE OF THE BENDING MOMENT AT THE BLADE TIP FOR SEVERAL VALUES OF THE CENTRIFUGAL FORCE (ACTING ON THE BLADES ONLY)

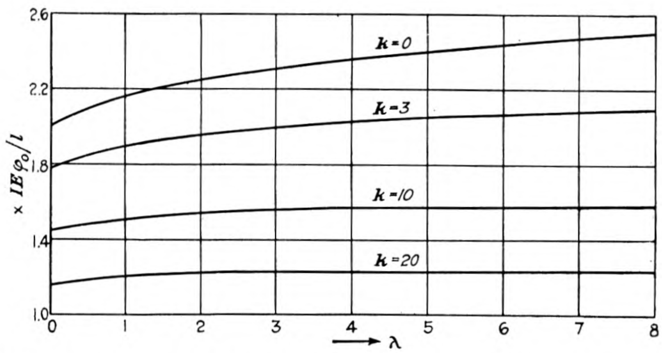

FIG. 14 GREATEST VALUE OF THE BENDING MOMENT AT THE ROOT FOR DIFFERENT VALUES OF THE CENTRIFUGAL FORCE (ACTING ON THE BLADE ONLY); UNIFORMLY DISTRIBUTED STEAM LOAD

segment of six blades, and the maximum bending moment in blades and wire was plotted as a function of the "ratio of flexibility" λ (Figs. 12, 13, and 14). This was done for different values of the factor k, which is directly proportional to the centrifugal force.

4—CENTRIFUGAL FORCE ACTING ON THE WIRE ONLY; THE BLADES ARE NOT LOADED

The case can again be worked out by difference calculus, and the results apply to any particular blade out of a segment of any number of blades. We have the difference equation:

$$\varphi_{i-1} + (4 + 2\lambda)\varphi_i + \varphi_{i+1} = 0$$

Observing boundary conditions and neglecting high powers of β leads to:

$$\varphi_i = \frac{-\,p^* t^{*4}}{24 I_s E_s t} \; \frac{(\beta^i - \beta^{n-i+1})}{1 + 2\beta}$$

$$M_i = \frac{p^* t^{*2} \lambda}{6(1 + 2\beta)} (\beta^i - \beta^{n-i+1}) \qquad M_i^* = \frac{M_i}{2}$$

$$M_i' = \frac{p^* t^{*2}}{12}\left(1 - \frac{2\beta^i - 2\beta^{n+1-i} + \beta^{i-1} - \beta^{n+2-i}}{1 + 2\beta}\right)$$

$$M_i'' = \frac{p^* t^{*2}}{12}\left(-1 - \frac{2\beta^i - 2\beta^{n+1-i} + \beta^{i+1} - \beta^{n-i}}{1 + 2\beta}\right)$$

$$1 < i < n : T_i = p^* t^* \left(1 - \frac{\beta^{i-1} - \beta^{i+1} + \beta^{n-i} - \beta^{n+2-i}}{4(1 + 2\beta)}\right)$$

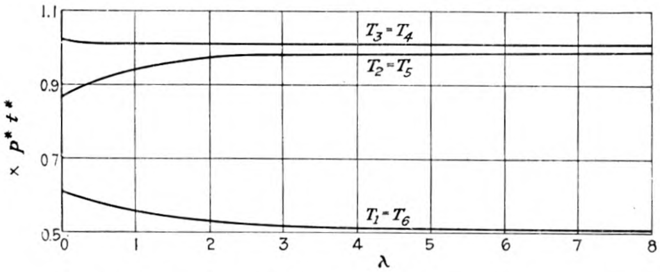

FIG. 15 PULL ON BLADES BY CENTRIFUGAL FORCE ON WIRE

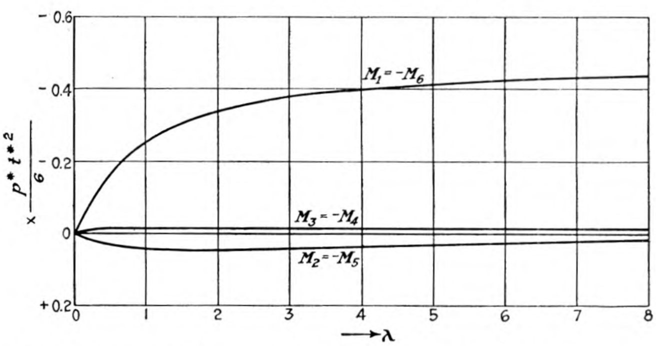

FIG. 16 BENDING MOMENTS ON BLADE TIPS DUE TO CENTRIFUGAL FORCE ON LASHING WIRE

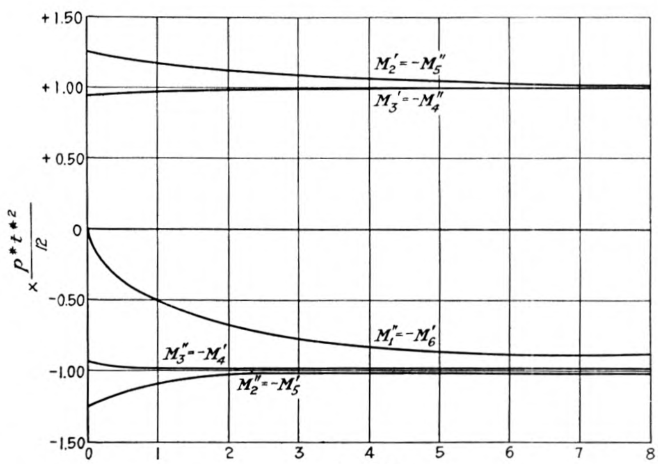

FIG. 17 BENDING MOMENTS IN LASHING WIRE DUE TO CENTRIFUGAL FORCE ON WIRE

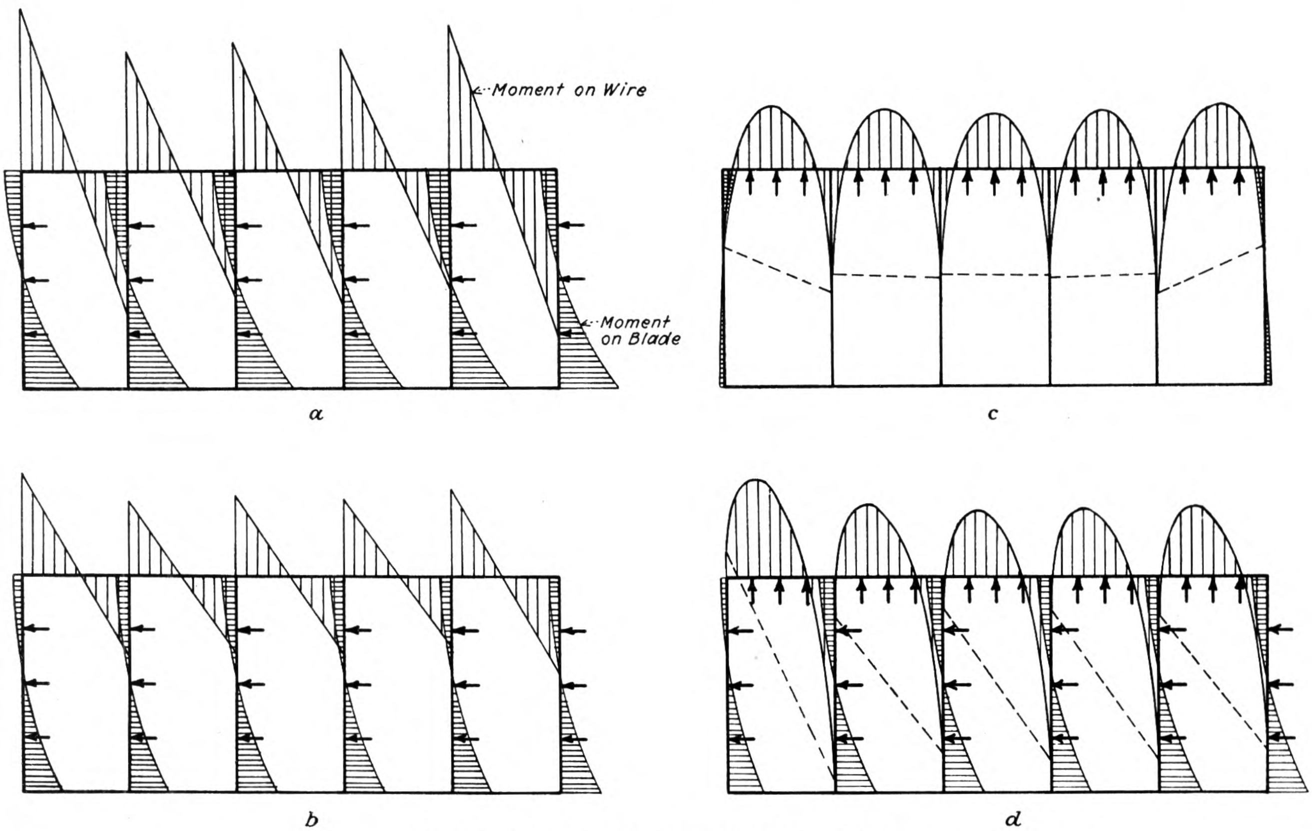

FIG. 18 BENDING MOMENTS FOR BLADES AND WIRE
(*a*, lateral load only; *b*, lateral load and centrifugal force on blades; *c*, centrifugal force on blades and wire; *d*, lateral load on blades, centrifugal force on blades and wire. Scale for moments on blades = 5 × scale for moments on wire.)

$$i = 1, i = n : T = p^*t^*\left(\frac{1}{2} + \frac{\beta^i + \beta^{i+1} - \beta^{n-i} - \beta^{n+1-i}}{4(1 + 2\beta)}\right)$$

These values are plotted again as a function of λ for a group of six blades (Figs. 15, 16, and 17). The bending stresses set up in the wire by centrifugal force are considerable for high speeds and large blade pitch.

5—Centrifugal Force on Lashing Wire and Blades; No Lateral Load on Blades

In Section 4 we gave results obtained for centrifugal force acting on the lashing wire only. It can be shown that the influence of centrifugal force on the blade can easily be taken into account by replacing λ by $\lambda^* = \dfrac{\lambda}{4\mu - \dfrac{3\rho^2}{\sigma}}$ where μ, ρ, and σ are correcting factors shown in Figs. 10 and 11. The value of β in the expressions corresponds now also to λ^* and not to λ. With these modifications, the formulas developed in Section 4 may readily be used except for the bending moment M_1^* at the root, which is now:

$$M_i^* = M_i\left(- \omega + \frac{3}{2}\frac{\rho\tau}{\sigma}\right)$$

6—Final Distribution

Actually, blades and lashing wire are subject to centrifugal forces and there is a steam load on the blade.

To give a complete picture of the distribution of the bending stresses, we have to superimpose the cases of Sections 3 and 5.

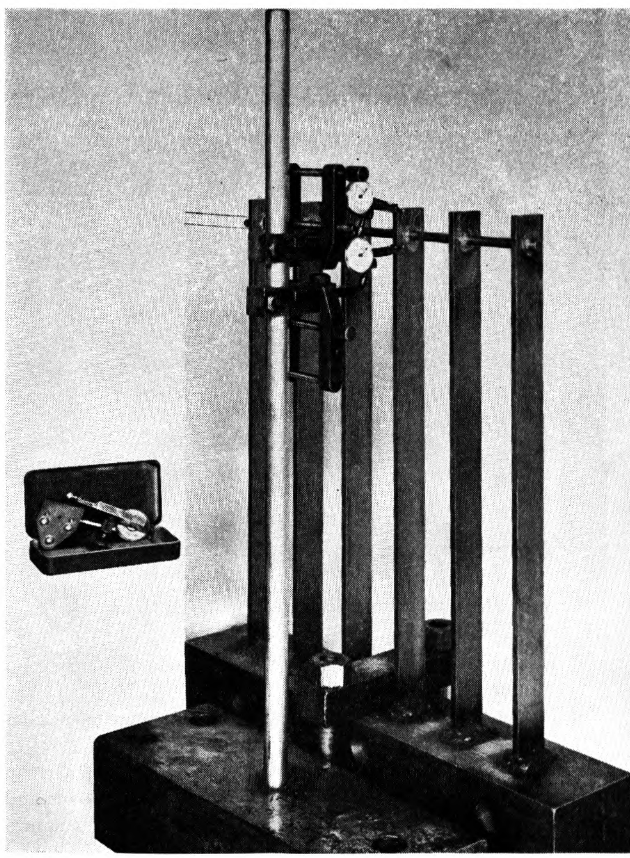

FIG. 19 MEASURING DEFLECTION AND TIP ANGLES
(Curvature instrument also shown.)

An example was worked out with the values:

$$\lambda = 1.84 \qquad k = 10 \qquad \frac{p^* t^{*2}}{12} = 20 \text{ lb-in.} \quad \frac{IE\varphi_0}{l} = 50 \text{ lb-in.}$$

Fig. 18 illustrates the distribution of the bending moments along wire and blades for the following cases:

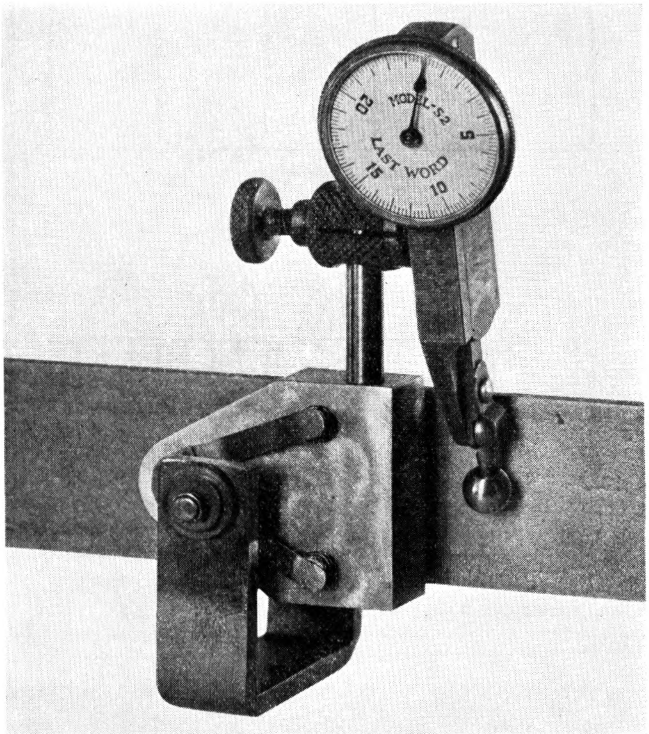

Fig. 20 Curvature Instrument Attached to Test Piece

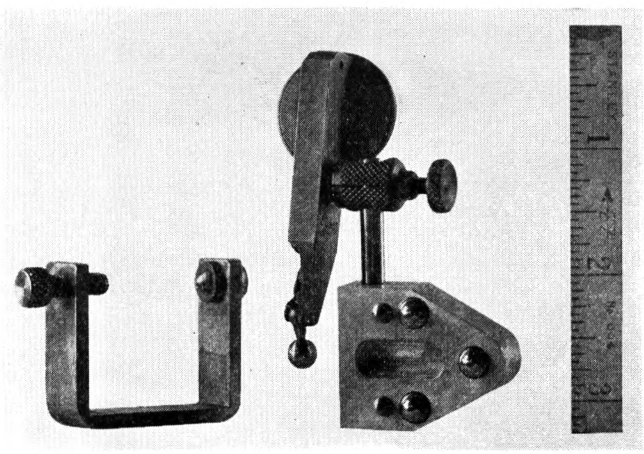

Fig. 21 Curvature Instrument With Clamp

(*a*) Blades under uniformly distributed load. No centrifugal force
(*b*) Blades under lateral load. Centrifugal force on blades only
(*c*) Centrifugal force on blades and wire. No lateral load
(*d*) Actual case: a blade group under steam load and centrifugal force. This figure is obtained by a superposition of (*b*) and (*c*).

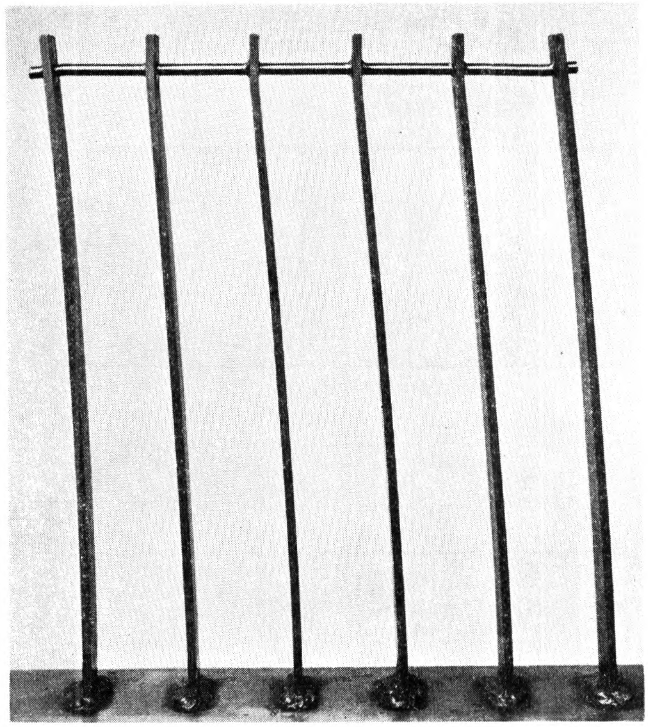

Fig. 22 Model of Blade Segment With Permanent Set

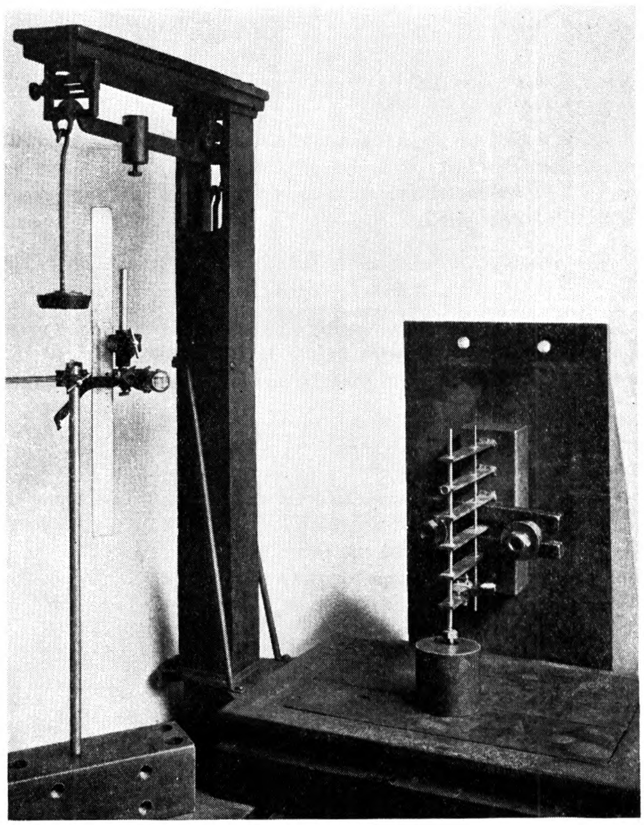

Fig. 23 Measuring Angles of Deflection With Telescope and Stresses With Curvature Instrument
(The load is read on the scale.)

Model Tests

The necessity was felt to check these purely theoretical computations with model tests. A photograph of the set-up is shown in Fig. 19. There was used a group of steel strips welded on a massive block. Two sizes of lashing wire were tested. The load was applied by pulling the wire; this gives the same force at the tip of each blade (considering bending deformations only). The test data checked nicely with the theoretical values, the maximum discrepancy being 7 per cent for the stresses and 5 per cent for the deflection at the lashing wire.

On model tests of this size there is often no room for standard tensometers. A small simple instrument, based on measuring the change in curvature, has rendered very good services in our case and may be worth while to be described (Figs. 20 and 21). Three steel balls are fitted into a small triangular plate and pressed against the test specimen (in our case an ordinary steel strip). Rigidly connected to the plate is a dial indicator provided with a small ball which rests against the specimen. The indicator follows every change in curvature of the strip, and the readings on the dial are directly proportional to the bending stress.

Two conical points are used to fix the location of the instrument. For tests which require only a short time, the triangular plate can be pressed by hand against the test piece; otherwise a clamping arrangement, like the one shown in Figs. 20 and 21, can be used. The clamp must have a flexible part, as it may not have a stiffening effect on the specimen. For this purpose we used a small rubber gasket.

Fig. 22 shows a model where permanent set is produced by pulling in the direction of the lashing wire. It is interesting to note how the end blades deflect differently from the others. Yielding is (according to the theory) produced in the wire next to the end blades, but practically not next to the middle blades. The middle blades have yielded at the tip, where the end blades remain straight.

Fig. 23 shows a more elaborate set-up used to measure stresses and deflections in a blade segment. The block with the strips is clamped against a vertical plate, and the load is applied at the lashing wire by adjusting a screw, which varies the distance between the lashing wire and the platform of a scale. The load is read directly on the scale. The angles of deflection are measured by a telescopic arrangement, a small mirror being fixed on the blades.

Acknowledgment

The author wishes to thank Dr. S. Timoshenko, whose encouragement has been most helpful in carrying out this investigation.

Appendix

THE INFLUENCE OF CENTRIFUGAL FORCE ON SINGLE BLADES WITH UNIFORMLY DISTRIBUTED LOAD q, TIP MOMENT M, AND TIP FORCE S

In the text we derived for constant section blades:

$$\frac{d^2u}{dz^2} = uz + \frac{S}{a} \qquad [8]$$

where $u = q + p\dfrac{dy_x}{dx}$

We put:

$$u = \sum_{i=0}^{\infty} A_i z^{3i} + \sum_{k=0}^{\infty} B_k z^{3k+1} + \sum_{n=0}^{\infty} C_n z^{3n+2}$$

Equation [8] is satisfied when: $A_{i+1} = \dfrac{A_i}{(3i+2)(3i+3)}$, $B_{k+1} = \dfrac{B_k}{(3k+3)(3k+4)}$, $C_{n+1} = \dfrac{C_n}{(3n+4)(3n+5)}$, where necessarily $C_0 = \dfrac{S}{2a}$.

Thus we find, observing boundary conditions:

$$u = A_0\left\{1 + \frac{z^3}{2\times 3} + \frac{z^6}{2\times 3\times 5\times 6} + \frac{z^9}{2\times 3\times 5\times 6\times 8\times 9}\cdots\right\}$$
$$+ \frac{M}{a^2}\left\{z + \frac{z^4}{3\times 4} + \frac{z^7}{3\times 4\times 6\times 7} + \frac{z^{10}}{3\times 4\times 6\times 7\times 9\times 10}\cdots\right\}$$
$$+ \frac{S}{2a}\left\{z^2 + \frac{z^5}{4\times 5} + \frac{z^8}{4\times 5\times 7\times 8} + \frac{z^{11}}{4\times 5\times 7\times 8\times 10\times 11}\cdots\right\}$$

(1) *Considering a uniformly distributed load q alone ($M = 0$, $S = 0$)*
At the blade root ($x = l$) the angle of rotation dy_x/dx is zero, thus $u = q = A_0\left\{1 + \dfrac{k}{2\times 3} + \dfrac{k^2}{2\times 3\times 5\times 6}\cdots\right\}$, where $k = \dfrac{l^3}{a^3} = \dfrac{pl^3}{I_bE_b}$ = directly proportional to the centrifugal force. This determines $A_0 = \dfrac{q}{1 + \dfrac{k}{2\times 3} + \dfrac{k^2}{2\times 3\times 5\times 6}}\cdots$

$$\text{so that } u = \frac{q\left\{1 + \dfrac{z^3}{2\times 3} + \dfrac{z^6}{2\times 3\times 5\times 6}\cdots\right\}}{1 + \dfrac{k}{2\times 3} + \dfrac{k^2}{2\times 3\times 5\times 6}\cdots} = q + p\frac{dy_x}{dx}$$

Expressions can now be derived for the deflection, the angle of rotation, and the bending moment at any point x. We are particularly interested in the maximum deflection y and angle dy_x/dx (at the tip) and the maximum bending moment $I_bE_b(d^2y_x/dx^2)$ (at the root). We find:

Maximum deflection y =

$$\frac{ql^4}{8I_bE_b}\,\frac{\left(1 + \dfrac{4k}{3\times 5\times 7} + \dfrac{4k^2}{3\times 5\times 6\times 8\times 10}\cdots\right)}{\left(1 + \dfrac{k}{2\times 3} + \dfrac{k^2}{2\times 3\times 5\times 6}\cdots\right)} = \delta\,\frac{ql^4}{8I_bE_b}$$

Maximum angle of rotation y' =

$$\frac{-ql^3}{6I_bE_b}\,\frac{\left(1 + \dfrac{k}{5\times 6} + \dfrac{k^2}{5\times 6\times 8\times 9}\cdots\right)}{\left(1 + \dfrac{k}{2\times 3} + \dfrac{k^2}{2\times 3\times 5\times 6}\cdots\right)} = -\epsilon\,\frac{ql^3}{6I_bE_b}$$

Maximum bending moment I_bE_by'' =

$$\frac{ql^2}{2}\,\frac{\left(1 + \dfrac{k}{2\times 3\times 5} + \dfrac{k^2}{3\times 5\times 6\times 8}\cdots\right)}{\left(1 + \dfrac{k}{2\times 3} + \dfrac{k^2}{2\times 3\times 5\times 6}\cdots\right)} = \gamma\,\frac{ql^2}{2}$$

Noting that for stationary blades and the same load $\dfrac{ql^4}{8I_bE_b}$ is the maximum deflection y_0, $-\dfrac{ql^3}{6I_bE_b}$ is the tip angle y_0', and $\dfrac{ql^2}{2}$ is the maximum bending moment $I_bE_by_0''$, we can write:

$$y = \delta y_0, \qquad y' = \epsilon y_0', \qquad I_bE_by'' = \gamma I_bE_by_0''$$

where δ, ϵ, and γ are correction factors taking into account the effect of centrifugal force. These factors are functions of k only (Fig. 9).

(2) *Considering a bending moment M (on the tip) alone.*

In a similar way as was done for uniformly distributed load, correction factors can be found which take into account the influence of the centrifugal force.

We may write:
For tip deflections: $y = \rho y_0$ (y_0 = without centrifugal force)
For tip angles: $y' = \mu y_0'$ (y_0' = without centrifugal force)
For root bending moments: $I_bE_by'' = \omega I_bE_by_0''$ ($I_bE_by_0''$ without centrifugal force)

$$\text{where } \rho = \frac{1 + \dfrac{k}{4\times 5} + \dfrac{k^2}{4\times 5\times 7\times 8}\cdots}{1 + \dfrac{k}{2\times 3} + \dfrac{k^2}{2\times 3\times 5\times 6}\cdots}$$

$$\mu = \frac{1 + \dfrac{k}{3\times 4} + \dfrac{k^2}{3\times 4\times 6\times 7}\cdots}{1 + \dfrac{k}{2\times 3} + \dfrac{k^2}{2\times 3\times 5\times 6}\cdots}$$

$$\omega = \frac{1}{1 + \dfrac{k}{2\times 3} + \dfrac{k^2}{2\times 3\times 5\times 6}\cdots}$$

ρ, μ, ω are plotted as a function of k in Fig. 10.

(3) *Considering the lateral force S on the tip alone.*

Again we introduce correction factors for the centrifugal force:
Tip deflections: $y = \sigma y_0$
Tip angles: $y' = \rho y_0'$
Root bending moments: $I_bE_by'' = \tau I_bE_by_0''$

$$\text{where } \sigma = \frac{1 + \dfrac{k}{4\times 6} + \dfrac{13k^2}{3\times 4\times 5\times 6\times 7\times 8}\cdots}{1 + \dfrac{k}{2\times 3} + \dfrac{k^2}{2\times 3\times 5\times 6}\cdots}$$

$$\rho = \frac{1 + \dfrac{k}{4 \times 5} + \dfrac{k^2}{4 \times 5 \times 7 \times 8} \cdots}{1 + \dfrac{k}{2 \times 3} + \dfrac{k^2}{2 \times 3 \times 5 \times 6} \cdots}$$

$$\tau = \frac{1 + \dfrac{k}{4 \times 6} + \dfrac{k^2}{4 \times 5 \times 7 \times 9} \cdots}{1 + \dfrac{k}{2 \times 3} + \dfrac{k^2}{2 \times 3 \times 5 \times 6} \cdots}$$

Curves for σ, ρ, and τ are shown in Fig. 11.

LIST OF NOTATIONS

l = length of blade (in.)
x = variable distance from tip (in.)
t = pitch on lashing wire diameter (in.)
t^* = length of lashing wire between two blades (in.)
I_b = minimum moment of inertia of blade (in.^{4})
E_b = modulus of elasticity of blade material (lb per sq in.)
I_s = moment of inertia of lashing wire (in.^{4})
E_s = modulus of elasticity of lashing material (lb per sq in.)
$\lambda = \dfrac{t^{*2} I_b E_b}{t l I_s E_s}$ = ratio of flexibility
$\beta = -(\lambda + 2) + \sqrt{(\lambda^2 + 4\lambda + 3)}$
n = number of blades
$D = \dfrac{2\lambda^2 + 26\lambda + 24}{3\lambda} + \dfrac{12 + \lambda}{3\lambda}\beta - \dfrac{6}{n(1 - \beta)}$
q = distributed lateral load on blade (lb per in.)
p = centrifugal load per unit length on blade (lb per in.)
p^* = centrifugal load per unit length on wire (lb per in.)
M (M_1, M_2, M_i) = bending moment (1st, 2d, ith blade) (lb-in.) on blade tip
M', (M_1', M_2', M_i') = bending moment in lashing wire for a cross-section just to the left of the 1st, 2d, ith blade (lb-in.)
M'' (M_1'', M_2'', M_i'') = bending moment in lashing wire for a cross-section just to the right of the 1st, 2d, ith blade (lb-in.)
M^* (M_1^*, M_2^*, M_i^*) = bending moment at blade root (1st, 2d, ith blade) (lb-in.)
S (S_1, S_2, S_i) = lateral shearing force at tip (lb)
T (T_1, T_2, T_i) = vertical force on tip (lb)
φ (φ_1, φ_2, φ_i) = angle of deflection at tip (radians)
y = lateral deflection (in.)
φ_0 = angle of deflection at tip without lashing wire ($= \dfrac{ql^3}{6 I_b E_b}$ for uniformly distributed load)
M_0^* = bending moment at root without lashing wire (lb-in.)

$$a^3 = \frac{I_b E_b}{p} \qquad z = \frac{x}{a} \qquad k = \frac{l^3}{a^3} = \frac{pl^3}{I_b E_b}$$