

A New Theory for the Buckling of Thin Cylinders Under Axial Compression and Bending¹

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The results of experiments on axial loading of cylindrical shells (thin enough to buckle below the elastic limit and too short to buckle as Euler columns) are not in good agreement with previous theories, which have been based on the assumptions of perfect initial shape and infinitesimal deflections. Experimental failure stresses range from 0.6 to 0.15 of the theoretical. The discrepancy is apparently considerably greater for brass and mild-steel specimens than for duralumin and increases with the radius-thickness ratio. There is an equally great discrepancy between observed and predicted shapes of buckling deflections.

In this paper an approximate large-deflection theory is developed, which permits initial eccentricities or deviations from cylindrical shape to be considered. True instability is, of course, impossible under such conditions; the stress distribution is no longer uniform, and it is assumed that final failure takes place when the maximum stress reaches the yield point. The effect of initial eccentricities and of large deflections is much greater than for the case of simple struts. Measurements of initial eccentricities in actual cylinders have not been made; however, it is shown that most of these discrepancies can be explained if the initial deviations from cylindrical form are assumed to be resolved into a double harmonic series, and

if certain reasonable assumptions are made as to the magnitudes of these components of the deviations. With these assumptions the failing stress is found to be a function of the yield point as well as of the modulus of elasticity and the radius-thickness ratio. On the basis of this a tentative design formula [5] is proposed, which involves relations suggested by the theory but is based on experimental data.

It is shown that similar discrepancies between experiments and previous theories on the buckling of thin cylinders in pure bending can be reasonably explained on the same basis, and that the maximum bending stress can be taken as about 1.4 times the values given by Equation [5]. It is also shown that puzzling features in many other buckling problems can probably be explained by similar considerations, and it is hoped that this discussion may help to open a new field in the study of buckling problems. The large-deflection theory developed in the paper should be useful in exploring this field, and may be used in other applications as well.

The paper presents the results of about a hundred new tests of thin cylinders in axial compression and bending, which, together with numerous tests by Lundquist,³ form the experimental evidence for the conclusions arrived at.

SYMBOLS USED

E, μ, σ_y = the modulus of elasticity, Poisson's ratio, and yield-point stress of the material

σ = average compressive stress in the axial direction, produced by the external load

r, t = mean radius and wall thickness of cylinder

x, s = axial and circumferential coordinates

u, v, w = axial, circumferential, and radial displacements of the middle surface of the wall as shown in Fig. 10.

w_1, w_2 = the initial radial displacement considered, and the

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NOTE: Statements and opinions advanced in papers are to be understood as individual expressions of their authors, and not those of the Society.

radial displacement under load ($w = w_2 - w_1$ is the movement due to the load)

W, W_1 = numbers proportional to the amplitudes of w and w_1

f = a stress function, defined by Equation [15]

L_x, L_s = wave-lengths of the deformation in the x and s directions

$\epsilon_x, \epsilon_s, \epsilon_{xs}, K_x, K_s, K_{xs}$ = extensional and flexural strains of the middle surface of the cylinder wall

$T_x, T_s, T_{xs}, G_x, G_s, G_{xs}$ = internal forces and moments per unit length of section as shown in Fig. 11

α, n = constants having to do with the assumed shape of the initial displacement, defined by Equation [4]

E = the internal strain energy due to the deformation

$c = \sqrt{12(1 - \mu^2)}$ (≈ 3.3 for engineering metals)

$P = c \frac{\sigma r}{E t}; \quad P_y = c \frac{\sigma_y r}{E t}$

$X = \frac{4\pi^2}{c} \frac{rt}{L_x^2}; \quad S = \frac{4\pi^2}{c} \frac{rt}{L_s^2}$

$K = 1 + 2 \frac{w_1}{w} = 1 + 2 \frac{W_1}{W}$

GENERAL DISCUSSION OF PROBLEM AND RESULTS

THIS paper applies to cylinders thin enough to buckle below the elastic limit, and too short to buckle first as Euler columns. Such cylinders, if carefully made and tested, buckle in small regular waves, as shown in Fig. 1.

³ N.A.C.A. Report No. 473.

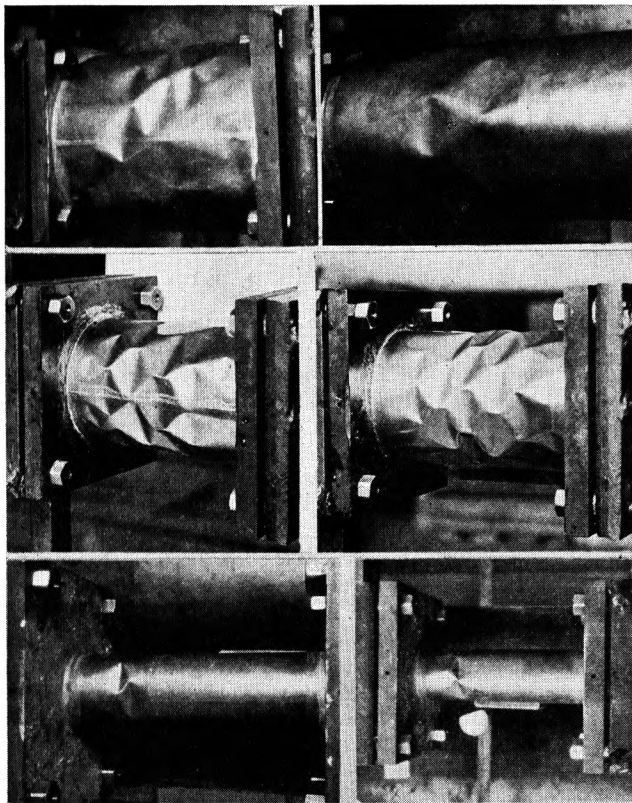


FIG. 1 TYPICAL FAILURES OF THIN CYLINDERS IN AXIAL COMPRESSION

In all the experiments cited in this paper the ends of the cylinders were clamped or fixed in some way. This stabilized the wall of the cylinder near the ends to such an extent that buckling always started at some distance from the ends. When cylinders are tested free-ended, eccentricity of loading and other local conditions at the ends are likely to obtain. Hence, buckling failure may take place at the ends at a lower load than would be required to buckle the main part of the cylinder. Such local effects are not important in most practical applications (as the ends are usually fastened with some degree of fixity) and will not be considered in this paper.

As is evident from the illustrations, the buckling waves observed in experiments are comparatively small. In most cases there were about ten waves around the circumference, and the wave-length in the axial direction was invariably of about the same size as in the circumferential direction. This immediately suggests that the length of the cylinder should have little effect on its buckling load unless it is very short (with a length less than one or two wave-lengths). This conclusion is entirely borne out by the tests. No correlation could be observed between the length and strength, although many series of cylinders, identical except for length, were tested. In a great many cases buckling occurred over only a comparatively small part of the length, the rest of the cylinder remaining entirely unbuckled. It is evident from these facts that, except for very short cylinders, the exact degree of end fixity (provided it is sufficient to insure against local weakness as already mentioned) can have little effect on the buckling strength, as different degrees of end fixity are roughly equivalent to different effective lengths. It will be assumed in this paper that the cylinders are long enough so that length and end conditions can be neglected. (The tests indicate that they can be neglected even when the length-radius ratio is considerably less than one.)

It is shown in Appendix 1 that, if length and end conditions are neglected, a theory based on the assumption of infinitesimal deflections and perfect initial shape leads to theoretical values for the buckling stress under axial load, and the wave-length of the buckling deflection given by the following:

$$P = 2 \dots \dots \dots [1]$$

$$\frac{(X + S)^2}{X} = 1 \dots \dots \dots [2]$$

These results were obtained by Robertson⁴ by a simplification of equations obtained by Southwell. The same results can be obtained by neglecting items which experiments show to be negligible in a still more complete solution obtained by Timoshenko.⁵ In Appendix 1 a much shorter derivation is given, based on simplified equilibrium equations developed by the author.⁶ The same results can also be obtained by energy considerations.

In Fig. 2 this theoretical value of P is compared with the values of P given by some hundred tests made by the author and by Lundquist.³ Numerous striking discrepancies will be noted:

- (a) There is a great scattering of the experimental points.
- (b) All the experimental values of P , and therefore of the buckling stress σ , are very much lower than the theoretical value.

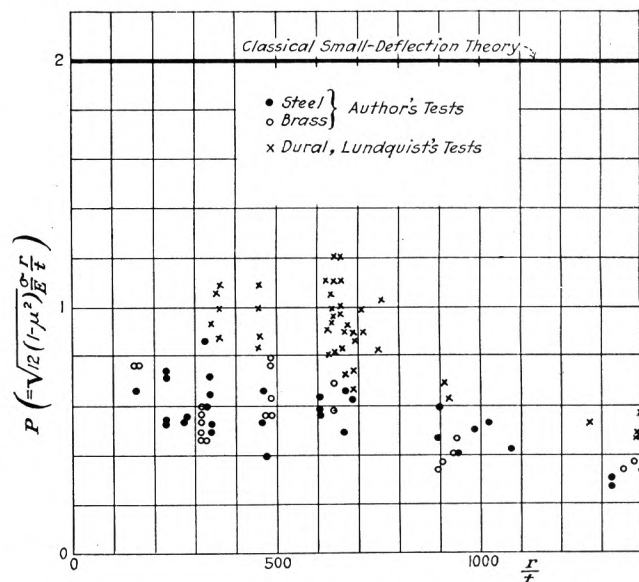


FIG. 2 COMPARISON OF EXPERIMENTAL STRENGTHS OF THIN CYLINDERS UNDER AXIAL COMPRESSION WITH CLASSICAL THEORY

- (c) Instead of being constant, the experimental values of P show a very decided tendency to become smaller with increasing r/l .

- (d) The experiments made by the author, which were made on brass and steel specimens, give consistently lower values of P than those made by Lundquist, which were made with duralumin specimens.

As to the shape of the buckling deflections, a first discrepancy is that in all experiments the wave-lengths in the axial and circumferential directions are consistently nearly equal, which would require that $X = S$, whereas Equation [2] requires no definite relation between X and S but only that $(X + S)^2/X$ have a definite value. Thus [2] would be satisfied, for instance, if

⁴ R. & M. No. 1185, British A.R.C., 1929.

⁵ S. Timoshenko, "Theory of Elasticity," 1914 (in Russian), p. 392.

⁶ N.A.C.A. Report No. 479.

$S = 0$, $X = 1$, in which case [1] and [2] reduce to the "symmetrical" theory for the buckling of cylinders under axial compression.⁷

If we assume that $X = S$, as indicated by the tests, [2] gives

$$X = S = 1/4 \dots \dots \dots [3]$$

Fig. 3 shows this value compared with the values found from the experiments. It will be seen that again there is great scattering, but that the experimental values of X or S are consistently much smaller than the theoretical; that is, the experimental wavelengths are much larger than the theoretical.

Attempts to explain these discrepancies have not been very satisfactory. The author believes that there is nothing incorrect in the classical analysis described, but that the assumption it makes that there are no initial deviations from true cylindrical shape (or unevennesses in the physical properties of the material or other imperfections—for the purposes of our discussion these can be assumed to be replaced by equivalent deviations from perfect shape) is not permissible in this case, at least for ordinary test specimens or ordinary practical applications. Theories neglecting initial inaccuracies, which are, of course, always present to some extent, give good approximations in other buckling problems, but these inaccuracies seem to be much more important in this case.

It is easy to explain why this is so. When a developable surface, such as a flat or a cylindrical surface, is deformed to a non-developable surface, there are produced (besides the flexural and stretching strains due to change of radius, which are considered in usual theories) strains which might be called "large-deflection strains," that result from stretching and compressing the developable shape into a non-developable one. These are of more importance than is commonly realized. In the bending of a strut a large-deflection theory is not needed unless the deflections are of the order of magnitude of the length of the strut, and as such deflections are of little practical interest, large-deflection theories have received little attention. But the aforementioned "large-deflection strains" in sheets become of importance in general when the deflections are of the order of magnitude of the *thickness* of the sheet.

For the very thin cylindrical shells which are under consideration (especially those rolled up from sheets, as were the specimens in the tests, and as is the case in all common applications), the initial deviations from cylindrical form are already of this order of magnitude. When the compressive load is applied these initial displacements are, of course, increased. The stresses due to the "large-deflection strains" accompanying this movement increase very rapidly, and combined with the direct compressive stress and the other stresses commonly considered, reach the yield point of the material at certain points long before the load has risen to the value given by Equation [1]. Beyond this point it is evident that the resistance of the cylinder will rapidly fall, so that complete failure must take place soon afterward.

It may be argued that the same effect must take place in the buckling of flat panels, and that, in this case, the ultimate failure is far *above* that given by the usual stability theory. This is true, but it does not invalidate the explanation given. There is nothing surprising in the fact that in one case the ultimate load—reached soon after the most highly stressed material passes the yield point—comes well *above* the classical stability limit, while in the other case it comes well *below*. In the case of a thin flat panel the stability limit itself is very low; the load at which the combination of common and "large-deflection stresses" reaches the yield point (at the edges of the panel) is much higher, and the panel can go through the stability limit without complete

failure, because of the artificial support given the edges. In the case of a cylinder the classical stability limit is comparatively very high, and the large-deflection stresses increase very fast owing to the small size of the waves, so that failure is brought on by yielding at a lower load than is indicated by the classical theory. But the two cases are similar in principle, and a complete solution of the problem of the ultimate strength of flat panels can only be obtained by using large-deflection theory.⁸ The simplified

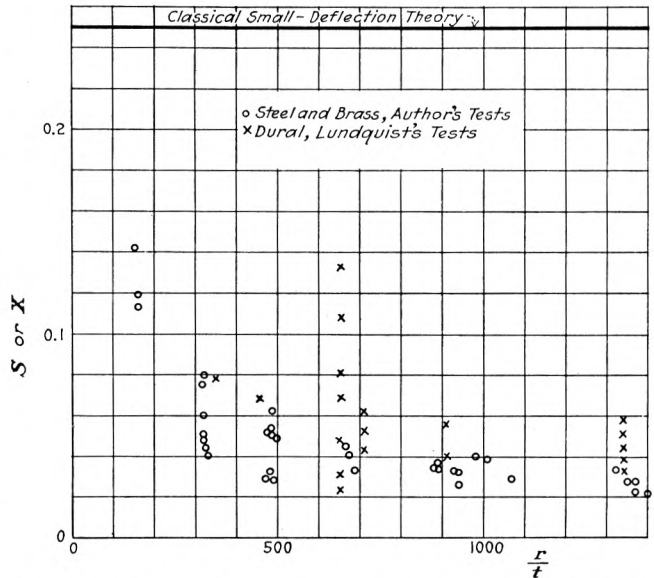


FIG. 3 COMPARISON OF SIZE OF BUCKLING WAVES AS GIVEN BY TESTS AND BY CLASSICAL THEORY

large-deflection theory developed in this paper should be useful in making such a study. The initial displacements are probably not important for this case, so that it should be possible to obtain a more definite result than in the problem of the present paper.

In other buckling problems, such as the buckling of struts and the buckling of thin cylinders under torsion, initial displacements or other inaccuracies must also be present, and yet in these cases a reasonable check with experiments is obtained without considering these questions. The explanation is that in these cases, as in the case of the flat panel, the classical stability limit is *below* the load at which the most highly stressed point would pass the yield point (because the large-deflection stresses are absent in the case of a strut, and they are less important in the case of torsion of a cylinder, as the buckling shape is more nearly a developable surface) and there is no artificial support which can carry some part of the structure through the general stability limit without failure, as is the case with the edges of flat panels. But there may be unexplored ranges of dimensions or materials where these questions are important, even in these cases. The author has noticed that in the buckling under torsion of very short cylinders, for which the buckling shape is further from a developable surface than it is for a long cylinder, the yield point is usually reached at about the theoretical stability limit. In this case the stiffening effect of the large-deflection stresses (and in general such stresses must always have a stiffening effect, because the increase in internal energy due to them must be supplied by an increase in the external load) just about balances the opposite effect of the yielding of the material, so that yielding and deepening of the buckling deflections continue for a long time with little change in

⁷ S. Timoshenko, *Zeitschrift für Mathematik und Physik*, vol. 58, 1910.

⁸ A very approximate solution for this problem has been given by von Kármán. See "The Strength of Thin Plates in Compression," A.S.M.E. Trans., vol. 54, 1932, paper APM-54-5.

the external load.⁹ We evidently have here a third type of problem in which these questions are important. It is thought that these few remarks, and the discussion of the case of the cylinder under axial compression, may help to clarify other points in the study of buckling problems, which hitherto have been puzzling.

Let us now consider in more detail just what happens when a compressive load is applied to a thin sheet or strut having initial deviations from straightness in the direction of the compression. Consider first the simple case of a hinged-end strut of a certain size. If it has an initial displacement in the shape of a half sine wave, the amplitude of the displacement will be increased when a compressive load is applied. The ratio of increase will be small for small loads, but will become very great when the load approaches the theoretical stability limit. If the initial displacement were in the shape of a full sine wave, the amplitude of this displacement would also increase when a compressive load is applied. The ratio of increase would also be small for small loads and would increase as the load increases. It would become infinite if the load could be raised to four times the theoretical stability limit. Now imagine that the initial displacement is a combination of the two displacements mentioned. If the displacement is small, the action which takes place will be a superposition of the two actions just described. The rates of increase of the two components will be enormously different when the load is near the theoretical stability limit, but they will not be so very different when the load is small.

Similar actions will take place in the case of a cylinder under axial compression. For convenience in discussing this question we may consider the total initial radial displacement from true cylindrical form to be made up of numerous component displacements, each of wave form (similar to the displacements shown in Fig. 1), but having different wave-lengths in the x , or s , or both directions. Now suppose that only one of these components of the displacement is present. As the compressive load is increased, the amplitude of this component will increase until, at some point of the wave, because of the combination of the direct stress (the average compressive stress in the axial direction resulting from the load) with the other stresses produced by the deformation, the yield point of the material is reached. Such yielding will occur simultaneously at corresponding points in each of the waves of the displacement all over the cylinder. When the load is increased beyond this point, it is obvious that the resistance to *this form* of displacement will be greatly lowered, and the displacement will increase very rapidly (with probably a falling resistance to the load, which would explain the sudden, explosive failure which always takes place in tests) until we have such a condition as is shown in Fig. 1.

Now let us suppose that all the components of the displacement are present. When a compressive load is applied to the cylinder the amplitudes of all the components will tend to increase. Stresses will be produced due to each one of these components. (It is shown later that the components can be defined so that the question of interaction between them is of minor importance; in any case stress systems having the wave patterns of each of the components will be produced—this is all that is necessary for the following argument.) *Combination* of the stresses produced by several different components, together with the direct stress, may cause yielding at certain points in the cylinder before yielding due to any *single* component, such as described in the last paragraph, takes place. But, due to the different wave-lengths of the different components, such yielding because of combinations of components will be local. It will not occur simultaneously at corresponding points in each of the waves of one of the components, as described before, and hence it will not produce a great weakening in the resistance to any one of the components.

⁹ N.A.C.A. Report No. 479, p. 12.

This probably explains another puzzling experimental fact, noted by Lundquist,¹⁰ which the author corroborates, that local buckling frequently occurs somewhere in the cylinder without precipitating complete failure, as the load can continue to increase without any general buckling until complete failure takes place suddenly over a large part of the cylinder at the load which would be expected if no preliminary buckling had taken place.

It now seems evident that final failure will, in general, take place when the stresses due to one of the components of the initial displacement (combined with the direct stress) reach the yield point. The component which produces final failure (and which evidently has the same wave-lengths as the final failure, Fig. 1) is thus the one which causes yielding to take place at the lowest value of the external load. We can determine the characteristics of this component from this fact by using minimum principles or their equivalent.

We would naturally expect that the component having wave-lengths as given by Equation [2] or [3] would cause this yielding, and hence final failure, sooner than any other component, as *its* amplitude will certainly increase faster than that of any other component, as the load is increased. But, as noted before, the tests indicate that the component actually causing failure always has a considerably longer wave-length than this. A reasonable explanation for this is that the initial amplitudes of the different components are not equal. It is certainly natural to suppose that the components of the initial radial displacements which have longer wave-lengths will, in general, have larger amplitudes as well. Hence, failure may be precipitated by a component with a longer wave-length than [3], rather than by the component with this wave-length, because the first has a much greater initial amplitude, in spite of the fact that the amplitude of the second increases faster, and in spite of the fact, also, that the stresses produced by the deformation decrease as the wave-length increases. The calculations given later in this paper show that this is easily possible. In this connection it should be remembered that failure takes place at only a fraction of the theoretical stability limit given by [1], at which time the component with wave-length [3] does not increase so very much faster than its rival components.

We have been speaking of "components" of the initial displacement of the cylinder without being very definite as to their shape. Actually this is, of course, as stated in the beginning, only a convenient concept for the purposes of the previous discussion. To put the question on a definite basis, actual measurements should be made of the initial displacements in many specimens. The author has not had the opportunity of making such measurements and has no data of this sort from other sources. In the absence of such data the only thing that can be done is to discover if there is a possibility of explaining the experimental facts on the basis of reasonable assumptions as to the initial displacement.

The initial displacement, whatever it may be, can be analyzed into a double harmonic series. When an external compressive load is applied, all the terms of the series will, in general, tend to increase in amplitude. But their effects will not be independent, as with similar harmonic terms in the case of a strut. It can be said with confidence that, in most cases, the action of one term will be nearly independent of the action of another, that is, the combined effect will be nearly the same as the sum of their effects when present separately, but that there will probably be certain groups of terms whose action will decidedly not be independent. These will be groups of terms which, taken together, form a shape which it is "easier" for the cylinder wall to deflect to than a single pure harmonic shape. We would expect such groups to consist of a primary term, which determines the wave-lengths of the group,

¹⁰ N.A.C.A. Report No. 473, pp. 5 and 14.

and higher harmonics, which modify the shape of the primary term to an "easier" shape; or combinations of such terms with a secondary term or terms which enable some of the large-deflection stresses to be annulled by stresses due to change of radius. (We shall use such a term in the calculations given later.)

Each of the "components" of the initial displacement in the previous discussion can be considered to represent one of these groups of harmonic terms. Since we try to include in each group all the terms whose actions greatly affect each other, and since it has been shown previously that it is not important to consider combinations of the stresses due to different groups, it seems that we shall make no very great error if we consider only one of the groups and neglect the effect of all the others. The one we consider will of course be the one which precipitates failure, chosen by the condition that it gives the lowest final failure load. Any conclusions which we draw from this calculation will be on the conservative side, because the chief contention to be made is that the low failure load found in experiments can be explained by reasonable initial displacements, and any parts of the initial displacement which we neglect could hardly have had any other effect than to reduce the calculated failure load.

In the calculations, which are given in detail in Appendix 2, it is assumed that the initial displacement and the final displacement, and hence also the movement, are of the same geometrical shape. The assumed shape of displacement consists of a primary term taken as a harmonic function of x and s , with the wave-lengths L_x and L_s , and the amplitude $W_1 t/c$ or Wt/c (for the initial displacement and the movement), and one secondary term designed to annul as much as possible of the large-deflection stresses with stresses due to change of radius. Fig. 12a shows the primary term. It is evident that at $p - p$ the material is on the average circumferentially stretched, while the material at $q - q$ is on the average circumferentially compressed, to have equilibrium in the circumferential direction. By superposing the symmetrical deformation shown at Fig. 12b of the proper amplitude, we annul this average circumferential stretching and compressing, and although we introduce a certain amount of bending in the longitudinal direction, the total internal energy is considerably reduced, that is, the addition of the symmetrical term makes it a much "easier" form. The use of this term simplifies the calculations, and trial shows it to be very effective in reducing the final failure load. Without it, it is impossible to explain at all the low values of failure loads found in experiments. It is, of course, quite probable that a refinement in the magnitude of this term, or the addition of other terms, would be found to be still more effective in reducing the final failure load. No attempt has been made to make such refinements because of the complexity it would involve.

As to our justification for assuming that the initial displacement and the movement will have the shape assumed, there is no doubt that the movement will take this shape, or a still "easier" one, if possible. And in neglecting possible refinements in the shape, the conclusions which we draw from the calculation will be on the conservative side for the same reasons as were given in a previous case. The amplitude of the secondary term is proportional to W^2 , that is, it is a second-order term, and its absolute magnitude, as given by the calculations, is always very small compared to the primary term, so that its presence would not be noticeable in tests.

Of course we have no right to assume that the two terms are present in the actual initial displacement with the relative magnitudes which we have assumed, although it is possible that the action of curving flat sheet into cylinders tends to bring this about. But the effective magnitude of the whole group will be some kind of average determined by the magnitudes of the terms actually present, and the assumption that the calculated proportions

are actually present should be sufficient for our purpose—to see if the required magnitudes are reasonable. It might be emphasized here that it is not expected that the assumptions made as to the nature of the initial displacements are anywhere near exact. It is only necessary, for our contention, that they represent average tendencies—the great observed scattering in the experimental results explains the wide deviations from the assumptions which must be expected.

In setting up the theory, a combination of the equilibrium and energy principles is used. Expressions for the extensional and flexural strains of the middle surface of the wall are first set up by adding terms describing the large-deflection strains to the usual expressions. Using the usual relations between the internal forces and the strains, the equilibrium equations of the elements of the cylinder wall in the axial and circumferential direction are set up, the same as in small-deflection theory. These enable a "stress function" to be used, and it is then possible to derive, first, a relation between the stress function and the radial displacement; and second, an expression for the internal elastic energy in terms of these two variables and the properties of the cylinder. If, now, we assume an expression for the radial displacement, we can obtain the corresponding expression for the stress function from the first relation, and with the aid of the second expression and the principle of virtual work we can obtain the external compressive load required to produce the displacement. Using the expression for the displacement already described, we obtain P (defining the external load) as a function of X , S , W_1 , and W .

With the assumed displacement and the expressions previously found for the internal forces, we now set up the condition for yielding at any point of the displacement wave by the maximum-shear-energy theory. With this expression it would be possible to determine the exact point in the wave at which yielding first takes place, by maximum-minimum principles. This would be a very complicated calculation, however, so instead it is assumed that yielding first takes place at the nodes of the waves, where trial indicates that the stress condition is at least approximately as severe as anywhere. With this assumption we obtain $P_y \left(= \frac{c\sigma_y}{E} \frac{r}{t} \right)$ as a function of X , S , W_1 , and W .

If, now, we knew the actual value of W_1 , we could eliminate W between the two relations and obtain P as a function of P_y , X , and S . We could then, by trial or by using minimum theory, determine the X and S which make P a minimum (which means determining that "component" of the deflection which precipitates failure, as we previously decided to do), and thus find P as a function of P_y , which means finding σ as a function of the properties of the cylinder E , r/t , and $E/c\sigma_y$.

As we do not know W_1 , we shall do the reverse of this and determine the magnitude of W_1 , which, by the above process, gives values of P as low as shown by the experiments, and then see if this value of W_1 is reasonable. However, although we do not know the absolute magnitude of W_1 we can say something as to its probable variation—or the probable "average tendencies" of its variation—with t , L_x , L_s , etc. Thus for the flat sheets from which cylinders are made we can certainly expect the initial displacement to decrease with increasing thickness and that components of the displacement of shorter wave-length will have smaller magnitudes. Thus in Fig. 13 we would not expect two sheets rolled by the same method, one thin and one thick, to have components of displacement of like wave-length with the same amplitudes, as at a and b , but would rather expect the amplitude to be smaller in the thicker sheet, as at c . And in Fig. 14 we would not expect components of different wave-length in the same sheet to have the same amplitude, as in a and b , but would expect a smaller amplitude for a shorter wave-length as at c .

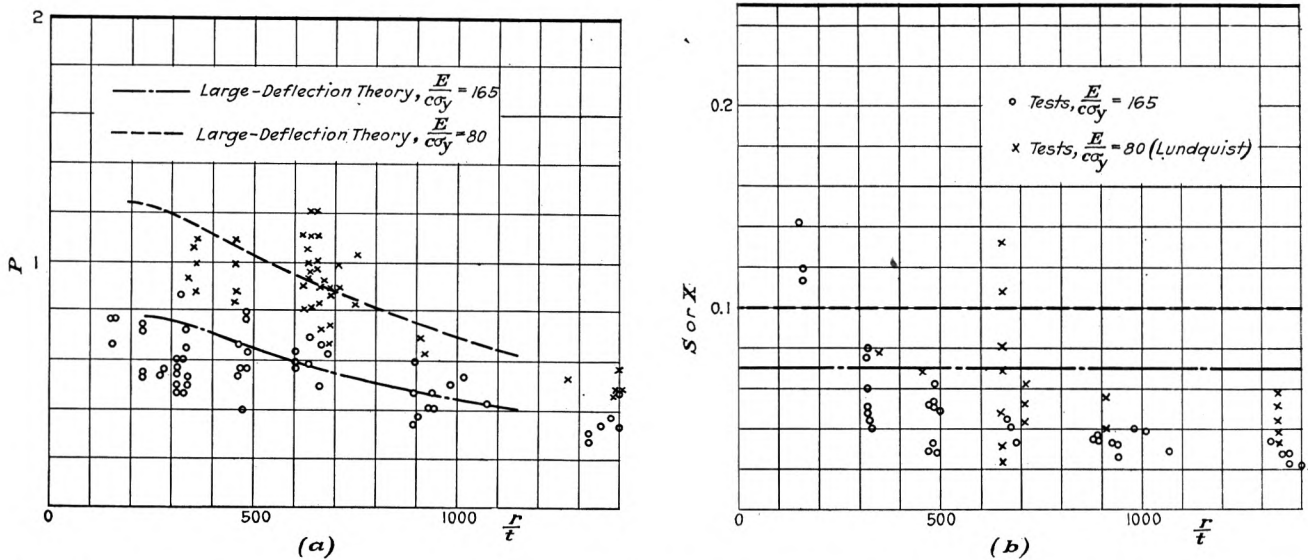


FIG. 4 DIAGRAMS SHOWING HOW EXPERIMENTAL RESULTS CAN BE EXPLAINED BY LARGE-DEFLECTION THEORY IF CERTAIN ASSUMPTIONS ARE MADE AS TO INITIAL DEVIATIONS FROM CYLINDRICAL FORM

(The theoretical value X or S could not be determined very exactly without a great deal of labor. Values shown are rough estimates.)

It is thus reasonable to expect that for the flat sheets W_1 would vary more or less as given by the equation

$$W_1 = \alpha \frac{L_z^n L_s^n}{t^{2n}} \dots \dots \dots [4]$$

where α and n are non-dimensional quantities more or less constant for sheets rolled by the same process.

The process of curving the sheet into cylinders will certainly change this expression for W_1 considerably, and doubtless introduce the quantity r into it. The nature of the change is something which could doubtless be analyzed, but a satisfactory analysis is a difficult problem in itself. An attempt was made to make a rough analysis, but the attempt was abandoned as it was felt that such results were very likely to be misleading. In the absence of a satisfactory solution, it was felt that much could still be learned by using [4] as it stands, as the argument on which it is based, already given, certainly holds for curved sheets as well as flat. Some discussion is given later of possible changes due to the process of curving the sheet. This question has no effect on the main contention of this paper, that the very low failure loads found in tests can be explained by reasonably small initial displacements.

A principal effect of curving the sheet into cylinders will probably be to upset the symmetry of expression [4] with respect to L_z and L_s . This will greatly affect the ratio between the values of L_z and L_s given by the theory. Without some knowledge of the nature of this effect it is useless to try to see if the theory will explain the fact that L_z and L_s are nearly equal in tests. Hence, the equality of L_z and L_s (and so of X and S) was assumed, to see if the other results of tests could be explained.

Using the expressions for P and P_y , already mentioned with [4], we find P as a function of $E/c\sigma_y$, r/t , X (or S), n , and α . As expected, it is found that the value of X (or S) which makes P a minimum depends on the value of n . It is found that we get about the value of X shown by tests if n is $5/4$, which is certainly a reasonable value. To bring P down to the level of the test values, α has to be about 1.1×10^{-5} . Using these values of n , X , and α , we obtain P as a function of $E/c\sigma_y$ and r/t .

The value of $E/c\sigma_y$ for the duralumin specimens tested by Lundquist was about 80, while the value of this quantity happened to

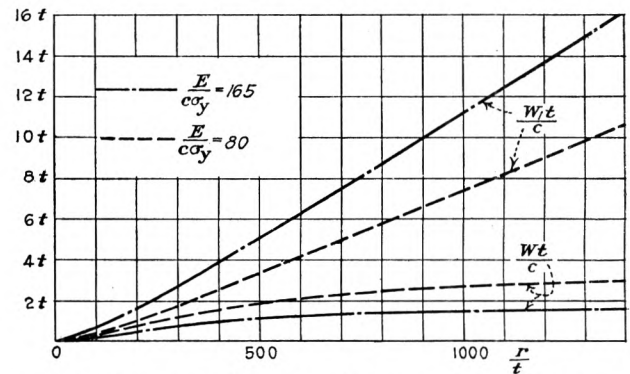


FIG. 5 VALUES OF Wt/c AND W_1t/c REQUIRED BY LARGE-DEFLECTION THEORY TO EXPLAIN TEST RESULTS

be about 165 for both the steel and the brass specimens tested by the author. For each of these values of $E/c\sigma_y$ we obtain P as a function of r/t . In Fig. 4a the corresponding values of P and r/t so obtained have been plotted, and the resulting curves can be compared with the experimental points for the same values of $E/c\sigma_y$. It will be seen that the accordance is excellent, the theoretical curves showing nearly the same downward slope with increase of r/t , and decrease of P with increasing $E/c\sigma_y$, as shown by the test results. In Fig. 4b the value of X or S corresponding to these results is compared with the test results.

Fig. 5 shows the values of W_1t/c , and the values of Wt/c at which failure starts, as given by the calculations. The values found for Wt/c , giving the magnitude of the movement under load up to the time that failure starts, are very small, which corresponds to the test experience that no general buckling can be noticed by eye up to the instant of the sudden failure. The values found for W_1t/c , which indicate the magnitude of the initial displacement which must be assumed to explain the low test failure loads, do not seem particularly excessive. They increase with the radius-thickness ratio, as common observation indicates they should, although this increase is due to the selection of a component with a longer wavelength, rather than any effect on the initial displacement by the curving of the sheet. Of course, these values represent only a part of the total initial displacement, but probably in practise

quite a large part; the fact that failure can apparently take place over only a part of the cylinder wall about as easily as over it all, as indicated by the tests, means that it is not necessary for the component most favorable to failure to be large or even to be present all over the cylinder wall, but failure can take place wherever, by accident, it happens to be large over a considerable portion of the wall. If measurements finally show that initial displacements are actually not as large as the theory calls for, then the relative crudity of the calculations is probably to blame. It will be remembered that most of the simplifications were on the conservative side in this respect. Since the crude secondary term used in the displacement had such an enormous effect in reducing the calculated values of P for a given value of W_1 (or reducing the value of W_1 for a given value of P), it is probable that further refinement would have an important effect in the same direction.

There is not much likelihood that the result obtained of checking the experimentally indicated decrease of P with increase of r/t and of E/σ_y is accidental and due only to the particular assumptions made. The author has tried many combinations and finds that any reasonable assumption regarding the initial displacement seems to result in the same general tendency. Thus, varying the value of n in [4] has little effect on the results except to change the value of X or S which gives the minimum P . One effect of curving flat sheets into cylinders is probably to reduce the size of initial unevennesses, so that we might expect some power of r/t in the numerator of [4], after this effect has been allowed for. Such an addition has also been tried, and it is found that we still get about the same reduction of P with increase of E/σ_y , and, as might be expected, still greater reduction of P with increase of r/t , a greater reduction than is indicated by experiments. However, the unknown effect of bending the sheets on the relation of W_1 to the wave-lengths may easily counterbalance this.

It is obvious that our assumptions as to the initial displacements, and to a less extent the calculations themselves, are at best only rough approximations—a fact which may be excused by the newness and difficulty of the problem. Much work, both theoretical and experimental, must be done before the question can be considered as settled. Other factors may enter the problem besides those which have been discussed. When this paper was presented recently at the Fourth International Congress of Applied Mechanics, Prof. R. V. Southwell made a very interesting suggestion. When a cylinder is compressed, the axial shortening is accompanied by a circumferential expansion. This expansion is largely prevented at the ends, where the cylinders are attached to something else, and this holding-in of the ends relative to the rest of the cylinder produces (since axial elements of the cylinder wall are in the condition of a beam on an elastic foundation) a symmetrical displacement, similar to that shown at Fig. 12b except that the amplitude is a maximum at the ends and damps out as we go toward the center. Calculations show that the wave-length of this displacement is the same as required for the secondary term of our assumed displacement when X or S is about 0.13. This value of X or S is about half that given by the classical stability theory, but is still not as small as the average of the tests calls for. This phenomenon undoubtedly takes some part in the buckling action but how important a part it takes is still to be determined. It is an experimental fact that when the buckling occurs over only a part of the length of the cylinder it usually occurs very near the end, where this displacement is a maximum. (We could not expect the buckling to occur still closer to the ends, on account of the fixity there.) Some of the photographs given show this. It is possible that this phenomenon may eliminate the necessity for some of the assumptions which have been proposed.

In spite of the roughness of the calculations, it is believed that

the most important factors have been taken into consideration and that the results, while they do not prove that the discrepancies between the experiments and the classical theory are to be explained in the general manner indicated, at least make this strongly probable.

It is particularly believed that it has been demonstrated that the dependence of P on r/t and E/σ_y , indicated by the tests, is not a mere accident, or explainable by variations in experimental technique. It therefore seems that this relation should be considered in design formulas. The most important practical signifi-

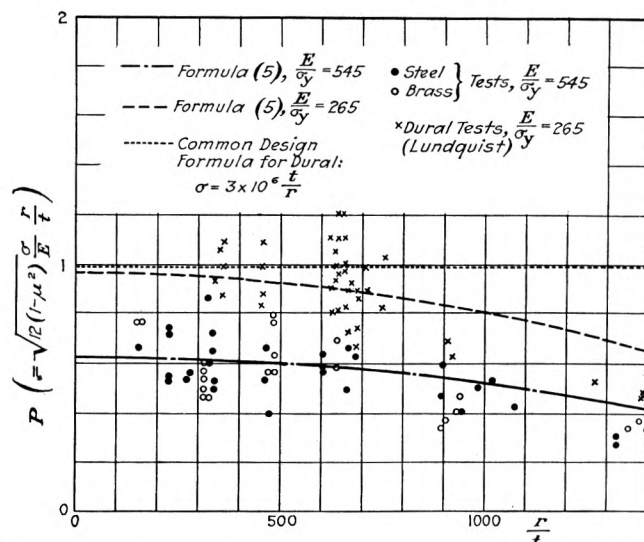


FIG. 6 COMPARISON OF EQUATION [5] WITH TEST RESULTS

cance of this relation is that some improvement in buckling strength can be allowed for if a material with a higher yield point is used, and vice versa. The following formula for the average failure stress

$$\sigma = E \frac{0.6 \frac{t}{r} - 10^{-7} \frac{r}{t}}{1 + 0.004 \frac{E}{\sigma_y}} \dots \dots \dots [5]$$

has been found to describe the relations shown by the tests quite well, and to give reasonable values for the extreme conditions $E/\sigma_y = 0$, $E/\sigma_y = \infty$, and $r/t = 0$. It gives a negative result when r/t is very large; in this case, which is far outside the practical range, σ can be taken as zero without great error. Curves obtained from this formula are compared with the test results in Fig. 6. As stated, the formula is designed to give the average strengths to be expected, and if it is desired to know the minimum strength likely to be encountered under any circumstances, some factor must be used with it; the value of the factor to be used under any given conditions can best be estimated from the test results shown in the figure. In the absence of anything better, this formula, with or without some factor, is recommended for design purposes.

No theoretical work has been done on the allied problem of the buckling of thin cylinders under pure bending. The smallness of the buckling waves found for axial loading immediately suggested that buckling will take place on the compression side of the bending specimen in practically the same way as it does in an axially loaded specimen, and that any results found for axial loading would apply also to pure bending, with some factor to allow for the fact that the stress varies from zero to a maximum instead of being constant around the circumference.

Numerous bending tests on specimens similar to those tested in axial compression completely confirmed this opinion. Photographs of typical specimens are shown in Fig. 7, while the results of the tests are plotted alongside the results obtained in axial compression, in Fig. 8. Buckling occurred over the compression side of the specimens in the same wave form, with approximately the same wave-lengths, as in the axially loaded specimens. It will be noticed from Fig. 8 that the results show exactly the same decrease of P with increase of r/t as shown by the axially

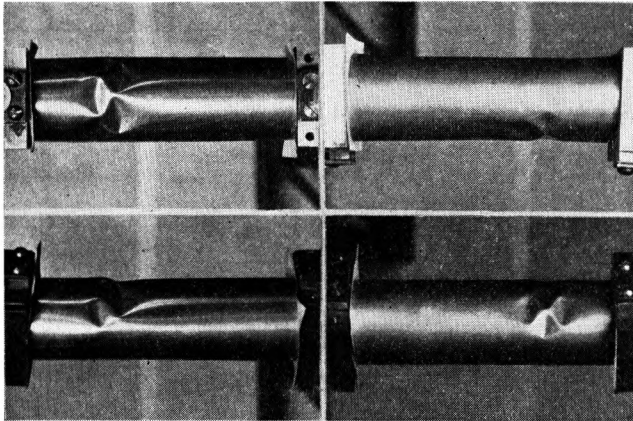


FIG. 7 TYPICAL FAILURES OF THIN CYLINDERS IN PURE BENDING

loaded specimens. In computing P the value of σ was taken as the maximum stress in bending, according to elementary theory. The values of P found are about 1.4 times the values found in axial-compression tests for all values of r/t . This is just about what would be expected, as it shows that general buckling takes place when the stress at a point in the cylinder wall about 45 deg to the neutral axis rises to the value which produces failure in a uniformly stressed specimen.

An ingenious theory for the stability of thin cylinders in bending has been advanced by Brazier.¹¹ According to this, the elastic curvature produced in the initially straight cylinder produces the well-known phenomenon of the flattening of the cross-sections of curved tubes under bending. The cross-section becomes more and more oval until a point is reached at which the resistance to bending starts to decrease, after which, of course, complete collapse takes place. Serious objection can be made to the theoretical derivation given by Brazier in that small-deflection theory is used and is assumed to apply after the deflections become very large; that is, the small-order terms neglected in the derivation (which can be neglected when the deflections are very small) are, at the critical point, of the same magnitude as the terms considered. However, it is an undoubted fact that this type of failure does take place in comparatively thick tubes made of a material with a low modulus, such as rubber tubes and thick metal tubes stressed above the yield point. It would seem that Brazier's type of failure and the small-wave type are more or less independent types of failure, and in an actual tube failure is produced by whichever type happens to occur first; that is, whichever type requires the least load. For thin metal tubes of the type considered in this paper, failure of the small-wave type, the same as in axial compression, probably always occurs first. For design purposes it seems safe to say that the maximum bending stress given by the elementary bending formula can rise to about 1.4 times the value given by [5] before buckling will, on the average, occur.

¹¹ R. & M. No. 1081, British A.R.C. 1926.

THE EXPERIMENTS

The results of the experiments have already been described. Detailed data are given in Tables 1 and 2 for the axial compression and pure bending tests respectively. The specimens were made in exactly the same way and tested on the same special testing machine (shown in Fig. 9) as the torsion specimens described in a previous paper by the author,⁶ and the reader is referred to this paper for a detailed description of the testing machine and the technique of making the specimens. The axially loaded specimens were loaded through a very frictionless universal joint very carefully centered to insure against eccentricity.

One fact, not previously mentioned, which was noticed in making the tests, is that no matter how quickly the loading was stopped after final failure took place it seemed impossible to reach as high a load on reloading as was reached the first time—and this in spite of the fact that the average compressive stress

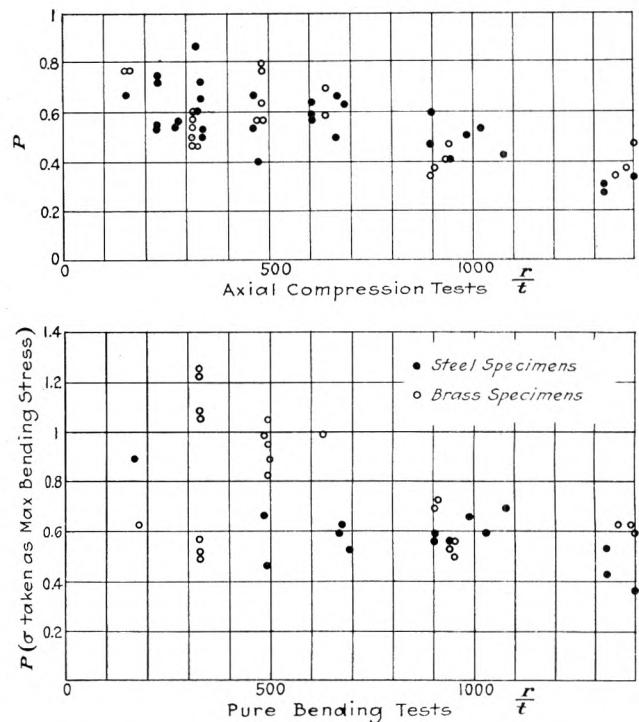


FIG. 8 COMPARISON OF AXIAL-COMPRESSION AND PURE-BENDING TESTS

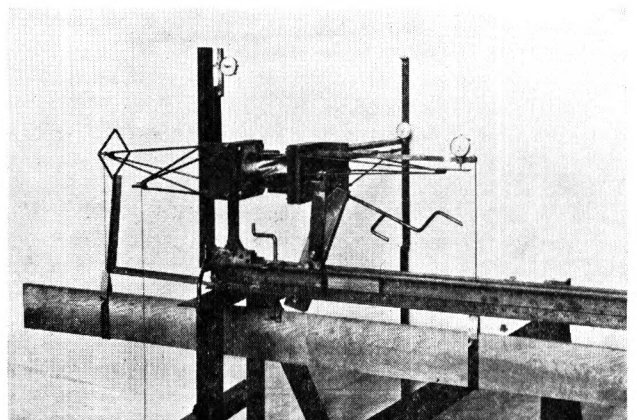


FIG. 9 SPECIAL TESTING MACHINE WHICH APPLIES ANY COMBINATION OF TORSION, BENDING, AND COMPRESSION (OR TENSION) LOADS

due to the load was in many cases only a small fraction of the yield point. This tends to bear out the contention that, even for very thin cylinders, the cause of final failure is the fact that at certain points, strategically located to weaken the cylinder for the type of failure involved, the local stress passes the yield point.

Another short series of tests which was made, the data for which are given in Table 3, bears out the same contention. In this eight cylinders were tested in axial compression. The cylinders were identical except that instead of being bent to the proper radius before making the joint, as was done with all the other specimens, some of them were bent to the proper radius, while others were sprung into shape from the flat sheet or from an entirely different radius. The results of these tests are shown in Fig. 15, in which the value of P obtained is plotted against the difference between the final and initial curvatures. It will be seen that the specimens with the highest initial stresses, due to the springing, consistently gave the lowest values of P . To be sure, the variation of P is within the limits of the scattering observed in other tests, so that this variation could possibly be accidental. This seems hardly likely, however, in view of the number of the tests and the fact that two specimens of each kind were tested and the same tendency was exhibited by both specimens. The fact that all specimens were cut from the same roll of material probably reduced the real scattering in this case.

These results have an important practical implication which is quite obvious. They also bear out the contention that final failure is precipitated by yielding of the material, as obviously the initial stresses due to springing the sheet (present on the outer and inner surfaces all over the cylinder) will combine with the stress due to other causes to produce yielding before it would otherwise occur. Of course, if the wall is bent to the proper shape before making the joint, some initial stresses will also be present, but they will be much smaller and more localized than where the sheet is sprung to shape. Also, the cold-working of the material may raise its yield point slightly, which would also fall in with our contention.

Appendix 1. Theory on the Assumption of Perfect Initial Shape and Infinitesimal Displacements

In a previous paper¹² the author has shown that the conditions for equilibrium of an element of the wall of a thin cylinder, under uniform axial compression, when the displacement consists of several waves around the circumference, can be simplified to

$$\frac{Et^2}{12(1-\mu^2)} \nabla^4 w + \frac{E}{r^2} \frac{\partial^4 w}{\partial x^4} = \sigma \nabla^4 \frac{\partial^2 w}{\partial x^2} \dots \dots \dots [6]$$

TABLE 1 AXIAL COMPRESSION TESTS¹³

Steel Tubes						Brass Tubes					
Diameter, (in.)	Length, (in.)	Thick-ness $\times 10^3$, (in.)	$E \times 10^{-6}$, (lb/in. ²)	Failing load, (lb)	Num-ber of waves in circum.	Diameter, (in.)	Length, (in.)	Thick-ness $\times 10^3$, (in.)	$E \times 10^{-6}$, (lb/in. ²)	Failing load, (lb)	Num-ber of waves in circum.
5.67	6	2.88	31.3	238	11	5.67	6	5.83	16.3	670	10
5.67	6	2.78	31.3	245	11	3.75	6	5.96	16.3	503	9
3.75	6	2.92	31.3	244	10	3.75	6	5.90	16.3	476	8
3.75	6	2.78	31.3	306	10	1.885	6	5.83	16.3	682	8
1.885	6	2.84	31.3	282	7	5.67	6	2.98	15.7	119	10
1.885	6	2.72	31.3	382	9	1.885	6	2.95	15.7	141	7
5.67	6	2.17	31.3	76	12	1.885	6	2.96	15.7	158	7
5.67	6	2.11	31.3	82	12	5.67	6	2.11	15.7	44	11
3.75	6	2.18	31.3	133	10	5.67	6	2.01	15.7	42	11
3.75	6	2.13	31.3	159	10	3.75	6	2.13	15.7	48	10
1.885	6	2.05	31.3	134	7	3.75	6	2.07	15.7	41	10
1.885	6	2.03	31.3	159	7	5.69	12	5.94	16.3	622	9
5.67	12	2.64	31.3	181	10	5.69	12	5.85	16.3	855	9
3.75	12	2.74	31.3	277	9	5.69	12	5.85	16.3	815	9
1.885	12	2.74	31.3	223	7	3.75	12	5.98	16.3	547	7
5.67	12	1.99	31.3	81	10	1.885	12	5.87	16.3	803	8
3.75	12	2.01	31.3	98	9	5.67	12	1.93	15.7	50	10
1.885	24	2.80	31.3	239	5	3.75	12	2.07	15.7	51	10
1.885	30	1.99	31.3	91	7	5.67	30	5.92	16.3	612	7
						3.75	30	5.91	16.3	612	7
						1.885	30	5.97	16.3	848	8

TABLE 2 PURE BENDING TESTS¹³

Steel Tubes						Brass Tubes					
Diameter, (in.)	Length, (in.)	Thick-ness, $\times 10^3$ (in.)	E $\times 10^{-6}$ (lb/in. ²)	Failing moment (in.-lb)	Num-ber of waves in circum.	Dia-me-ter, (in.)	Length, (in.)	Thick-ness, $\times 10^3$ (in.)	E $\times 10^{-6}$ (lb/in. ²)	Failing moment (in.-lb)	Num-ber of waves in circum.
5.67	6	2.86	31.3	458	11	5.67	6	5.81	16.3	1320	10
5.67	6	2.78	31.3	382	11	5.67	6	5.89	16.3	1500	11
3.75	6	2.88	31.3	270	11	3.75	6	5.89	16.3	1108	9
3.75	6	2.80	31.3	276	10	3.75	6	5.89	16.3	1222	8
5.67	6	2.13	31.3	162	12	1.885	6	5.96	16.3	322	8
5.67	6	2.09	31.3	196	12	5.67	6	3.03	15.7	198	10
3.75	6	2.13	31.3	150	10	5.67	6	3.09	15.7	220	10
3.75	6	2.11	31.3	136	11	3.75	6	3.00	15.7	252	10
5.67	12	2.70	31.3	428	9	1.885	6	2.97	15.7	156	5
3.75	12	2.74	31.3	224	8	5.67	6	2.13	15.7	122	11
5.67	12	1.99	31.3	122	10	5.67	6	2.05	15.7	114	11
3.75	12	1.99	31.3	126	10	3.75	6	2.09	15.7	88	10
1.885	24	1.99	31.3	72	6	3.75	6	2.09	15.7	86	10
						5.69	12	6.03	16.3	1284	8
						5.69	12	5.85	16.3	1462	..
						5.69	12	5.85	16.3	1570	..
						3.75	12	5.96	16.3	1118	9
						5.67	12	1.89	15.7	92	11
						3.75	12	2.11	15.7	64	9
						5.67	30	5.93	16.3	692	9
						1.885	24	5.95	16.3	468	8

TABLE 3 TESTS TO DETERMINE EFFECT OF INITIAL STRESSES ON FINAL COMPRESSIVE STRENGTH¹³

Original radius of curvature of sheet, in.	Failing load, lb	Original radius of curvature of sheet, in.	Failing load, lb
-0.56 ^a	125	1.5	180
-0.56 ^a	122	1.5	152
-4.3 ^a	157	0.56	119
-5.0 ^a	142	0.56	123

All cylinders were of brass, with diameter 3.75 in., length 6.0 in., thickness 0.00295 in., and E 15,700,000 lb per sq. in.
^a Negative sign indicates original curvature was in opposite direction from final curvature.

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2}$; ∇^4 signifies the application of ∇^2 twice; and ∇^8 four times. This equation is satisfied if

$$w = W \sin \frac{2\pi x}{L_x} \sin \frac{2\pi s}{L_s} \dots \dots \dots [7]$$

We neglect edge conditions entirely because, due to the small size of the waves, it is not important whether there are an even number of waves in the circumference, or what the conditions are at the ends of the cylinders. (In the tests, buckling frequently occurred over only a part of the length or circumference.) Substituting [7] in [6] and using the symbols of the present paper, we obtain

$$P = \frac{(X + S)^2}{X} + \frac{X}{(X + S)^2} \dots \dots \dots [8]$$

Equation [8] gives the values of P required to maintain various states of equilibrium involving different values of X and S .

¹³ σ_y for the steel used was around 57,000 lb per sq. in. in all tests.
 σ_y for the brass was between 28,000 and 30,000 lb per sq. in. in all tests.

Buckling will take place as soon as P rises to the lowest of these values. By inspection, or using minimum principles, the lowest value of P obtainable from [8] is 2, obtained when the quantity $(X + S)/X = 1$. We thus obtain Equations [1] and [2], which have been discussed.

Appendix 2. Theory Considering Initial Displacements and Finite Deflections

The strains of the middle surface of the cylinder wall are obtained in terms of the displacements u, v, w_1, w_2 from the geometrical relationships between them. We find, for the linear strains in the x and s directions, the shear strain, the changes in curvatures in the x and s directions, and the unit twist

$$\left. \begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w_2}{\partial x} \right)^2 - \frac{1}{2} \left(\frac{\partial w_1}{\partial x} \right)^2 = \frac{\partial u}{\partial x} + \frac{K}{2} \left(\frac{\partial w}{\partial x} \right)^2 \\ \epsilon_s &= \frac{\partial v}{\partial s} + \frac{w_2}{r} - \frac{w_1}{r} + \frac{1}{2} \left(\frac{\partial w_2}{\partial s} \right)^2 - \frac{1}{2} \left(\frac{\partial w_1}{\partial s} \right)^2 = \frac{\partial v}{\partial s} + \frac{w}{r} + \frac{K}{2} \left(\frac{\partial w}{\partial s} \right)^2 \\ \epsilon_{xs} &= \frac{\partial u}{\partial s} + \frac{\partial v}{\partial x} + \frac{\partial w_2}{\partial x} \frac{\partial w_2}{\partial s} - \frac{\partial w_1}{\partial x} \frac{\partial w_1}{\partial s} = \frac{\partial u}{\partial s} + \frac{\partial v}{\partial x} + K \frac{\partial w}{\partial x} \frac{\partial w}{\partial s} \\ \kappa_x &= \frac{\partial^2 w_2}{\partial x^2} - \frac{\partial^2 w_1}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} \\ \kappa_s &= \frac{\partial^2 w_2}{\partial s^2} - \frac{\partial^2 w_1}{\partial s^2} = \frac{\partial^2 w}{\partial s^2} \\ \kappa_{xs} &= \frac{\partial^2 w_2}{\partial x \partial s} - \frac{\partial^2 w_1}{\partial x \partial s} = \frac{\partial^2 w}{\partial x \partial s} \end{aligned} \right\} \dots\dots\dots [9]$$

The terms involving squares or products of derivatives are large-deflection strains representing the change in length of elements due to their slope. The other terms are the same as in the small-deflection theory.¹³ The second expressions for ϵ_x, κ_x , etc. are obtained by substituting the relations

$$w_1 = \frac{K-1}{2} w; \quad w_2 = w_1 + w = \frac{K+1}{2} w \dots\dots\dots [11]$$

in the first expressions, remembering that $K (= 1 + 2 \frac{w_1}{w})$ by definition) is a constant with respect to x and s , since w and w_1 are assumed to have the same geometrical form.

The relations between the strains and the internal stresses, given by Hooke's law, are the same as used in the small-deflection theory.¹² The internal forces and moments per unit length of section (see Fig. 11) are

$$\left. \begin{aligned} T_x &= \frac{Et}{1-\mu^2} (\epsilon_x + \mu \epsilon_s) \\ T_s &= \frac{Et}{1-\mu^2} (\epsilon_s + \mu \epsilon_x) \\ T_{xs} &= \frac{Et}{2(1+\mu)} \epsilon_{xs} \end{aligned} \right\} \dots\dots\dots [12]$$

$$\left. \begin{aligned} G_x &= \frac{Et^3}{c^2} (\kappa_x + \mu \kappa_s) \\ G_s &= \frac{Et^3}{c^2} (\kappa_s + \mu \kappa_x) \\ G_{xs} &= \frac{Et^3}{12(1+\mu)} \kappa_{xs} \end{aligned} \right\} \dots\dots\dots [13]$$

The equations of equilibrium of an element in the x and s directions can be taken the same as in the small-deflection theory,¹² because the only new internal forces which we are considering (which are not considered in the small-deflection theory) are the large-deflection stresses, which form a part of T_x, T_s , and T_{xs} . And T_x, T_s , and T_{xs} are fully considered in these equations:

$$\left. \begin{aligned} \Sigma F_x &= 0; \quad \frac{\partial T_x}{\partial x} + \frac{\partial T_{xs}}{\partial s} = 0 \\ \Sigma F_s &= 0; \quad \frac{\partial T_s}{\partial s} + \frac{\partial T_{xs}}{\partial x} = 0 \end{aligned} \right\} \dots\dots\dots [14]$$

These equations will be satisfied if we take

$$T_x = \frac{Et^3}{c} \frac{\partial^2 f}{\partial s^2}; \quad T_s = \frac{Et^3}{c} \frac{\partial^2 f}{\partial x^2}; \quad T_{xs} = -\frac{Et^3}{c} \frac{\partial^2 f}{\partial x \partial s} \dots [15]$$

where f is the usual stress function, or Airy function, except for the constant factor Et^3/c , which is used to simplify the results.

Equating the expressions for T_x, T_s , and T_{xs} in [12] and in [15] and solving for ϵ_x, ϵ_s , and ϵ_{xs} , we find

$$\left. \begin{aligned} \epsilon_x &= \frac{t}{c} \left(\frac{\partial^2 f}{\partial s^2} - \mu \frac{\partial^2 f}{\partial x^2} \right) \\ \epsilon_s &= \frac{t}{c} \left(\frac{\partial^2 f}{\partial x^2} - \mu \frac{\partial^2 f}{\partial s^2} \right) \\ \epsilon_{xs} &= -2(1+\mu) \frac{t}{c} \frac{\partial^2 f}{\partial x \partial s} \end{aligned} \right\} \dots\dots\dots [16]$$

We next eliminate u and v between the three equations of [9] by applying the operator $\frac{\partial^2}{\partial s^2}$ to the first equation, $\frac{\partial^2}{\partial x^2}$ to the second equation, and subtracting these two equations from the third equation, to which the operator $\frac{\partial^2}{\partial x \partial s}$ has been applied.

This gives us

$$\frac{\partial^2 \epsilon_x}{\partial s^2} + \frac{\partial^2 \epsilon_s}{\partial x^2} - \frac{\partial^2 \epsilon_{xs}}{\partial x \partial s} = \frac{1}{r} \frac{\partial^2 w}{\partial x^2} + K \left[\left(\frac{\partial^2 w}{\partial x \partial s} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial s^2} \right] \dots [17]$$

Substituting [16] in this, we obtain the following relation between the stress function f and the radial movement w

$$\frac{t}{c} \nabla^4 f = \frac{1}{r} \frac{\partial^2 w}{\partial x^2} + K \left[\left(\frac{\partial^2 w}{\partial x \partial s} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial s^2} \right] \dots\dots\dots [18]$$

An expression analogous to [18], for the case of a flat plate without initial displacement ($r = \infty, K = 1$) was first obtained by von Kármán.¹⁴ (However, the author made this derivation independently, before learning of von Kármán's solution.)

The internal elastic energy is

$$E = \frac{1}{2} \iint dx ds (T_x \epsilon_x + T_s \epsilon_s + T_{xs} \epsilon_{xs} + G_x \kappa_x + G_s \kappa_s + 2G_{xs} \kappa_{xs}) \dots\dots\dots [19]$$

Substituting Equations [15], [16], [13], and [10] in Equation [19] we obtain an expression for the internal elastic energy in terms of f and w . If f and w are harmonic functions of x and s , this simplifies to

$$E = \frac{Et^3}{2c^2} \iint dx ds [(\nabla^2 f)^2 + (\nabla^2 w)^2] \dots\dots\dots [20]$$

¹⁴ Enzyklopädie der Math. Wiss., vol. 4, art. 27.

Equations [18] and [20] are general formulas which can be used in solving many other large-deflection problems in which the initial displacement can be taken as geometrically similar to the final displacement, or as zero (in which case $K = 1$). For flat sheets the first term on the right-hand side of [18] drops out. If an approximate expression for w is assumed, an expression for f can be derived from [18] and the boundary or other conditions, after which [20] can be used to apply the principle of virtual work.

We assume for w the shape discussed before

$$w = \frac{t}{c} \left(W \sin \frac{2\pi x}{L_x} \sin \frac{2\pi s}{L_s} + \frac{KS}{8} W^2 \cos \frac{4\pi x}{L_x} \right) \dots [21]$$

and take $w_1 = \frac{W_1}{W} w$, as implied in the definition of K . Substituting [21] in [18] and using the symbols X and S , we find

$$\nabla^2 f = -\frac{cX}{r^2 t} W \sin \frac{2\pi x}{L_x} \sin \frac{2\pi s}{L_s} + \frac{cKXS}{2r^2 t} W^2 \cos \frac{4\pi x}{L_x} \dots [22]$$

From our knowledge of the physics of the problem we know that T_x , T_s , and T_{xs} will be harmonic functions, except for a constant component of T_x equal to $-\tau\sigma$. Hence, from [22], f can be taken as

$$= -\frac{tX}{c(X+S)^2} W \sin \frac{2\pi x}{L_x} \sin \frac{2\pi s}{L_s} + \frac{tKX}{32cS} W^2 \cos \frac{4\pi x}{L_x} + Cs^2 \dots [23]$$

The coefficients of the terms in [23] were found by taking them as unknowns, substituting [23] in [22], and solving for the values of the coefficients satisfying [22] for any values of x or s . The coefficient of the second term in [21] was determined in a similar way, so as to satisfy the condition that the part of T_x (found by using [23] in [15]) independent of s shall vanish, as discussed in the first part of the paper. The constant C in [23] is found from the condition that the constant part of T_x shall equal $-\tau\sigma$ or

$$\int_0^{2\pi r} T_x ds = -2\pi r t \sigma, \text{ from which}$$

$$C = -\frac{c\sigma}{2Et} = -\frac{P}{2r} \dots [24]$$

Using [21], [23], and [24] in [20] and integrating over the circumference and length, we find the internal elastic energy

$$E = \frac{\pi E t^3 h}{4c^2 r} W^2 \left[\frac{X^2}{(X+S)^2} + (X+S)^2 + X^2 K^2 W^2 \left(\frac{1}{32} + \frac{S^2}{2} \right) \right] + \frac{\pi h r \sigma^2}{E} \dots [25]$$

where h is the length of the cylinder, or of the part of the cylinder considered. The last term in [25] evidently represents the elastic work due to the ordinary elastic shortening of the cylinder under the load. This has no effect in our problem and the derivation could have been simplified by omitting the non-harmonic part of [23] which produces this term. However, the justification for such an omission might not have been clear.

The work done by the external forces during a virtual displacement dW is equal to $2\pi r t \sigma$ times the average distance the cylinder is shortened during such a displacement, or

$$\int_0^{2\pi r} t \sigma ds dW \frac{\partial}{\partial W} \left[\int_0^h \frac{K}{2} \left(\frac{\partial w}{\partial x} \right)^2 dx \right] \dots [26]$$

By the principle of virtual work, this can be equated to the change of E during the displacement dW which is $\frac{\partial E}{\partial w} dW$. When

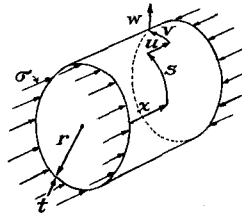


FIG. 10

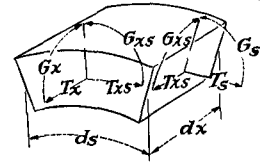


FIG. 11

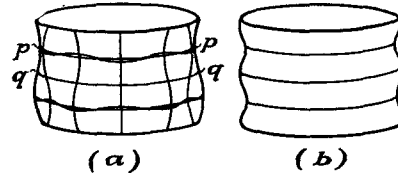


FIG. 12

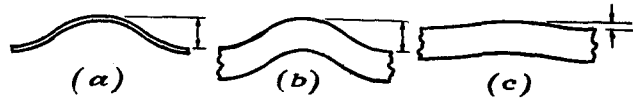


FIG. 13

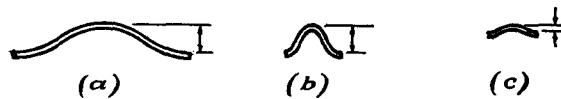


FIG. 14

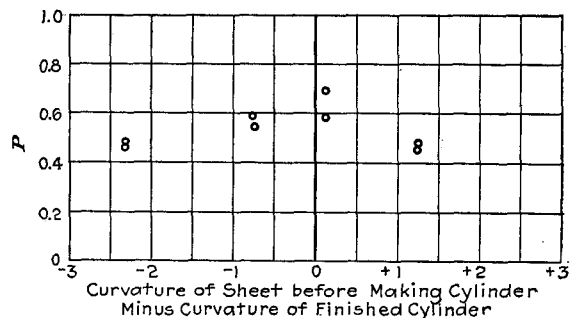


FIG. 15 RESULTS OF TESTS OF CYLINDERS WITH DIFFERENT INITIAL STRESSES DUE TO FORMING

these operations are carried out, after substituting [21] in [26] and remembering that K is a function of W , we obtain an expression for P in terms of W , W_1 , X , and S

$$P = \frac{\frac{(X+S)^2}{X} + \frac{X}{(X+S)^2} + KXW(W+W_1) \left(\frac{1}{16} + S^2 \right)}{\frac{W+W_1}{W} + \frac{KS^2W}{8} (2W+W_1)} \dots [27]$$

It will be observed that if W_1 is set equal to zero and terms containing W^2 are neglected as second-order terms, [27] reduces to Equation [8] of the classical theory.

In carrying out the integrations of [25] and [26] it is assumed that L_x and L_s are even multiples of the length and the circumference. This involves little error because of the small size of L_x and L_s , as discussed in the main part of this paper.

We shall now set up the condition for yielding of the material at any point. According to the maximum-shear-energy theory¹⁵ which is generally considered to be the most exact expression of

¹⁵ See "Plasticity," by A. Nadai, McGraw-Hill, N. Y., 1931.

the condition for yielding, plastic flow under combined stresses at any point commences when

$$(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 = 2\sigma_y^2 \dots [28]$$

where σ_1 , σ_2 , and σ_3 are the principal stresses at the point and σ_y is the yield-point stress in simple tension. In our case the stress in the radial direction can be taken as zero and as one of the principal stresses, while the other two principal stresses, in the plane of the cylinder wall, are given by the formula

$$\frac{1}{2} \left[(\sigma_x + \sigma_s) \pm \sqrt{(\sigma_x - \sigma_s)^2 + 4\sigma_{xs}^2} \right] \dots [29]$$

where σ_x , σ_s , and σ_{xs} are the normal and shear stresses on planes

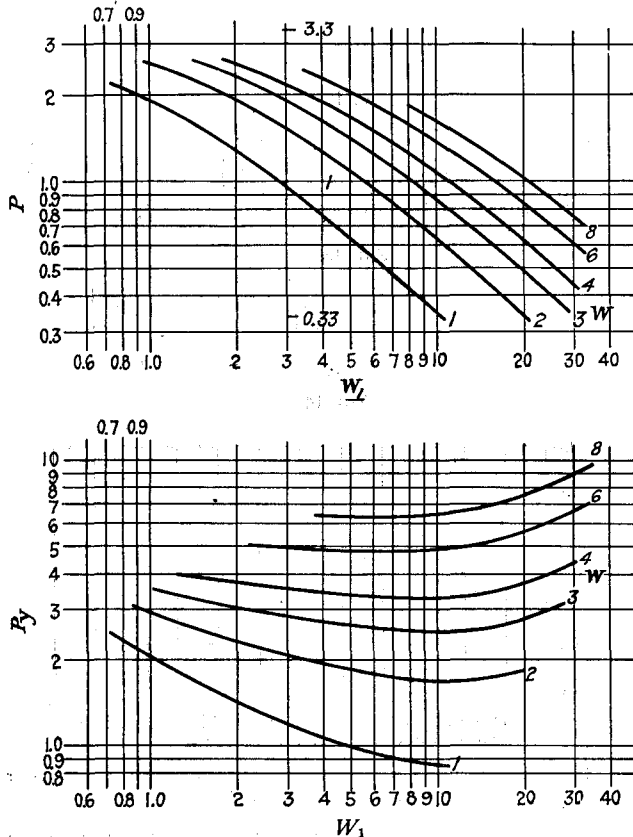


FIG. 16 THEORETICAL RELATION BETWEEN P , P_y , W_1 , AND W , FOR $X = S = 0.07$

perpendicular to the x and s directions. Using these values for the principal stresses in [28], we find

$$\sigma_x^2 - \sigma_x \sigma_s + \sigma_s^2 + 3\sigma_{xs}^2 = \sigma_y^2 \dots [30]$$

The values of σ_x , σ_s , and σ_{xs} at a point in the cylinder wall a distance z from the middle plane are, assuming a linear distribution of stress,

$$\sigma_x = \frac{T_x}{t} + \frac{12G_x}{t^3} z; \quad \sigma_s = \frac{T_s}{t} + \frac{12G_s}{t^3} z; \quad \sigma_{xs} = \frac{T_{xs}}{t} + \frac{12G_{xs}}{t^3} z [31]$$

Substituting [31] in [30] and using the expressions for T_x , G_x etc. [15], [13], [10], and finally [21], [23], and [24], we obtain an expression for P in terms of P_y , W , W_1 , X , S , x , s , and z . By using minimum theory we could determine the value of x , s , and z at which P is a minimum—that is, the point at which yielding first occurs, at the lowest value of σ . But the expression is too complex to make this very practical. Trials and elementary

calculations indicate that the material at the surface of the cylinder wall in the nodes of the wave form is in about as unfavorable a state as any, in most cases, at least. If we take $x = s = 0$ and $z = t/2$ in this expression we obtain

$$P_y^2 = P^2 + PKXW^2 \left(\frac{1}{4} + \frac{3(2-\mu)}{c} S \right) + 3XSW^2 \left(\frac{X}{(X+S)^2} + \frac{6(1-\mu)}{c} \right)^2 + (KXW^2)^2 \left(\frac{1}{64} + \frac{3(2-\mu)}{8c} S + \frac{9(1-\mu+\mu^2)}{c^2} S^2 \right) \dots [32]$$

Equation [4], discussed in the first part of the paper, can now be combined with Equations [27] and [32] to eliminate W and W_1 , and obtain P as a function of P_y , μ , X , S , n , and α . The influence of μ , which enters [32], is not important and a value of 0.3 can be taken for it for all engineering metals. We shall also assume that $X = S$ because of the difficulty of checking this experimentally verified relation, as previously discussed. With these assumptions [27] and [32] are somewhat simplified

$$P = W \frac{32X + \frac{2}{X} + KXW(W + W_1) \left(\frac{1}{2} + 8X^2 \right)}{8(W + W_1) + K^2X^2W^2(2W + W_1)} \dots [27']$$

$$P_y^2 = P^2 + PKXW^2 \left(\frac{1}{4} + 1.54X \right) + 3W^2 \left(\frac{1}{4} + 1.27X \right)^2 + (KXW^2)^2 \left(\frac{1}{64} + 0.193X + 0.65X^2 \right) \dots [32']$$

and Equation [4] becomes

$$W_1 = \alpha \left(\frac{12}{X} \frac{r}{t} \right)^n \dots [4']$$

Combining [27'], [32'], and [4'], we can obtain P as a function of P_y , r/t , X , n , and α . Then we can determine n so as to make the value of X , at which P is a minimum, coincide with test results, and determine α to bring the general magnitude of the values of P down to the level of test results.

The complexity of the equations made it impractical to do this directly. Actually, various values of W , W_1 , and X were assumed, which enabled the corresponding values of P and P_y to be found from [27'] and [32']. This gave sufficient data to plot

families of curves giving the relation of P and $P_y = \left(\frac{r}{t} \frac{E}{\sigma_y} \right)$ with W and W_1 for several values of X . Fig. 16 shows such families of curves for $X = 0.07$, and similar families were drawn for $X = 0.04$ and $X = 0.10$. Then, assuming values of α , n , X , and r/t , we find the value W_1 from [4']. Taking $E/\sigma_y = 165$, as in the tests made by the author, P_y can be calculated, and the corresponding values of W and P found from the curves such as shown in Fig. 16. Then by comparing the different results obtained for the different values of X , and plotting P against X for the same values of r/t , we can roughly determine the value of X at which P is a minimum. This gives sufficient data to plot curves such as shown in Figs. 4 and 5. This process was repeated with different values of α and n until the combination used in plotting Figs. 4 and 5 was found.

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